On the Low-SNR Capacity of Phase-Shift Keying with Hard-Decision Detection

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Abstract— [1] **The low-SNR capacity of M-ary PSK transmission over both the additive white Gaussian noise (AWGN) and fading channels is analyzed when hard-decision detection is employed at the receiver. Closed-form expressions for the first and second derivatives of the capacity at zero SNR are obtained. The spectral-efficiency/bit-energy tradeoff in the low-SNR regime is analyzed by finding the wideband slope and the bit energy required at zero spectral efficiency. Practical design guidelines are drawn from the information-theoretic analysis. The fading channel analysis is conducted for both coherent and noncoherent cases, and the performance penalty in the low-power regime for not knowing the channel is identified.**

I. Introduction

Phase modulation is a widely used technique for information transmission, and the performance of coded phase modulation has been of interest in the research community since the 1960s. One of the early works was conducted in [4] where the capacity and error exponents of a continuous-phase modulated system, in which the transmitted phase can assume any value in $[-\pi, \pi)$, is studied. More recent studies include [1], [2], [5], [6], [7], and [8]. Kaplan and Shamai studied in [5] the achievable information rates of differential phase-shift keying (DPSK) while reference [6] investigated the capacity of M-ary PSK over an additive white Gaussian noise (AWGN) channel with unknown phase that stays constant for a block of symbols. Pierce in [1] considered hard-decision detection of PSK signals trasmitted over the AWGN channel and compared the performances of 2-, 3- and 4-phase modulations. Pierce also provided in [1] an expression for the bit energy required by M-ary PSK at zero spectral efficiency. The authors in [2] analyzed the spectral efficiency of coded PSK and DPSK with soft- and hard-decision detection. Reference [7] analyzed the energy efficiency of PSK when it is combined with on-off keying for transmission over noncoherent Rician fading channels. Recent work by Zhang and Laneman [8] investigated the achievable rates of PSK over noncoherent Rayleigh fading channels with memory.

The low-SNR capacity of PSK with soft detection is well-understood. For instance, Verdú [3] has shown that quaternary PSK (QPSK) transmission over the AWGN or coherent fading channels is optimally efficient in the low-SNR regime, achieving both the minimum bit energy of -1.59 dB and optimal wideband slope which is defined as the slope of the spectral efficiency curve at zero spectral efficiency. Although soft detection gives the best performance, hard-decision detection and decoding is preferred when reduction in the computational burden is required [10]. Such a requirement, for instance, may be enforced in sensor networks [9]. Moreover, at very high transmission rates such as in fiber optic communications, obtaining multiple-bit resolution from A/D converters may not be possible [2]. Finally, it is of interest to understand the fundamental limits of hard-decision detection so that the performance gains of soft detection can be identified and weighed with its increased complexity requirements. Motivated by these considerations and the fact that the performance difference of hard and soft detections are more emphasized at low power levels, we study in this paper the low-SNR capacity of M-ary PSK over both the AWGN and fading channels when a hard-decision detection is employed at the receiver end.

II. Channel Model

We consider the following channel model

$$r_k = h_k s_{x_k} + n_k \quad k = 1, 2, 3 \ldots \tag{1}$$

where x_k is the discrete input, s_{x_k} is the transmitted signal when the input is x_k, and r_k is the received signal during the the k^{th} symbol duration. h_k is the channel gain. h_k is a fixed constant in unfaded AWGN channels, while in flat fading channels, h_k denotes the fading coefficient. $\{n_k\}$ is a sequence of independent and identically distributed (i.i.d.) zero-mean circularly symmetric Gaussian random variables denoting the additive background noise. The variance of n_k is $E\{|n_k|^2\} = N_0$. We assume that the system has an average power constraint of $E\{|s_{x_k}|^2\} \leq \mathcal{E} \quad \forall k$.

At the transmitter, M-ary PSK modulation is employed for transmission. Hence, the discrete input, x_k, takes values from $\{0, 1, \ldots, M-1\}$, and if $x_k = m$, then the transmitted signal in the k^{th} symbol duration is

$$s_{x_k} = s_m = \sqrt{\mathcal{E}} e^{j\theta_m} \tag{2}$$

where $\theta_m = \frac{2\pi m}{M} \quad m = 0, 1, \ldots, M-1$, is one of the M phases available in the constellation.

At the receiver, the detector makes hard decisions for every received symbol. Therefore, each received signal r_k is mapped to one of the points in the constellation set $\{\sqrt{\mathcal{E}} e^{j2\pi m/M}, m = 0, 1, \ldots, M-1\}$ before the decoding step. We assume that maximum likelihood decision rule is used at the detector. Note that with hard-decision detection, the channel can be

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now regarded as a symmetric discrete channel with M inputs and M outputs.

III. PSK over AWGN Channels

We first consider the unfaded AWGN channel and assume that $h = 1$. In this case, the conditional probability density function of the channel output given the channel input is[2]

$$f_{r|x}(r|x=m) = f_{r|s_m}(r|s_m) \quad (3)$$
$$= \frac{1}{\pi N_0} e^{-\frac{|r-s_m|^2}{N_0}} \quad m = 0, \ldots, M-1. \quad (4)$$

It is well-known that the maximum likelihood detector selects the constellation point closest to the received signal r. We denote the signal at the output of the detector by y and assume that $y \in \{0, 1, \ldots, M-1\}$. Note that $y = l$ for $l = 0, 1, \ldots, M-1$ means that the detected signal is $\sqrt{\mathcal{E}} e^{j2\pi l/M}$. The decision region for $y = l$ is the two-dimensional region

$$D_l = \left\{ r = |r|e^{j\theta} : \frac{(2l-1)\pi}{M} \le \theta < \frac{(2l+1)\pi}{M} \right\}. \quad (5)$$

With quantization at the receiver, the resulting channel is a symmetric, discrete, memoryless channel with input $x \in \{0, 1, \ldots, M-1\}$ and output $y \in \{0, 1, \ldots, M-1\}$. The transition probabilities are given by

$$P_{l,m} = P(y = l | x = m) \quad (6)$$
$$= P\left(\frac{(2l-1)\pi}{M} \le \theta < \frac{(2l+1)\pi}{M} | x = m \right) \quad (7)$$
$$= \int_{\frac{(2l-1)\pi}{M}}^{\frac{(2l+1)\pi}{M}} f_{\theta|s_m}(\theta|s_m)\, d\theta \quad (8)$$

where $f_{\theta|s_m}(\theta|s_m)$ is the conditional probability density function of the phase of the received signal given that the input is $x = m$, and hence the transmitted signal is s_m. It is well-known that the capacity of this symmetric channel is achieved by equiprobable inputs and the resulting capacity expression [11] is

$$C_M(\text{SNR}) = \log M - H(y|x=0) \quad (9)$$
$$= \log M + \sum_{l=0}^{M-1} P_{l,0} \log P_{l,0} \quad (10)$$

where $\text{SNR} = \frac{\mathcal{E}}{N_0}$, $H(\cdot)$ is the entropy function, and $P_{l,0} = P(y = l | x = 0)$. In order to evaluate the capacity of general M-ary PSK transmission with a hard-decision detector, the transition probabilities $\{P_{l,0}\}$ should be computed. Starting from (4), we can easily find that

$$f_{\theta|s_0}(\theta|s_0) = \frac{1}{2\pi} e^{-\text{SNR}} + \sqrt{\frac{\text{SNR}}{\pi}} \cos\theta \, e^{-\text{SNR}\sin^2\theta} \times \left(1 - Q(\sqrt{2\text{SNR}\cos^2\theta})\right) \quad (11)$$

where

$$Q(x) = \int_x^\infty \frac{1}{\sqrt{2\pi}} e^{-t^2/2}\, dt. \quad (12)$$

[2] Since the channel is memoryless, we henceforth, without loss of generality, drop the time index k in the equations for the sake of simplification.

Since $f_{\theta|s_0}$ is rather complicated, closed-form expressions for the capacity is available only for the special cases of $M = 2$ and 4:

$$C_2(\text{SNR}) = \log 2 - h(Q(\sqrt{2\text{SNR}})) \text{ and } C_4(\text{SNR}) = 2C_2\left(\frac{\text{SNR}}{2}\right)$$

where $h(x) = -x\log x - (1-x)\log(1-x)$. For the other cases, the channel capacity can only be found by numerical computation.

On the other hand, the behavior of the capacity in the low-SNR regime can be accurately predicted through the second-order Taylor series expansion of the capacity, which involves $\dot{C}_M(0)$ and $\ddot{C}_M(0)$, the first and second derivatives of the channel capacity (in nats/symbol) with respect to SNR at $\text{SNR} = 0$. In the following result, we provide closed-form expressions for these derivatives.

Theorem 1: The first and second derivatives of $C_M(\text{SNR})$ in nats per symbol at $\text{SNR} = 0$ are given by

$$\dot{C}_M(0) = \begin{cases} \frac{2}{\pi} & M = 2 \\ \frac{M^2}{4\pi}\sin^2\frac{\pi}{M} & M \ge 3 \end{cases}, \quad (13)$$

and

$$\ddot{C}_M(0) = \begin{cases} \frac{8}{3\pi}\left(\frac{1}{\pi} - 1\right) & M = 2 \\ \infty & M = 3 \\ \frac{4}{3\pi}\left(\frac{1}{\pi} - 1\right) & M = 4 \\ \psi(M) & M \ge 5 \end{cases} \quad (14)$$

respectively, where

$$\psi(M) = \frac{M^2}{16\pi^2}\left((2-\pi)\sin^2\frac{2\pi}{M} + (M^2 - 4\pi)\sin^4\frac{\pi}{M} - 2M\sin^2\frac{\pi}{M}\sin\frac{2\pi}{M} \right). \quad (15)$$

Proof: The main approach is to obtain $\dot{C}_M(0)$ and $\ddot{C}_M(0)$ by first finding the derivatives of the transition probabilities $\{P_{l,0}\}$. This can be accomplished by finding the first and second derivatives of $f_{\theta|s_0}$ with respect to SNR. However, the presence of $\sqrt{\text{SNR}}$ in second part of (11) complicates this approach because $\left.\frac{df_{\theta|s_0}}{d\text{SNR}}\right|_{\text{SNR}=0} = \infty$. In order to circumvent this problem, we define the new variable $a = \sqrt{\text{SNR}}$ and consider

$$f_{\theta|s_0}(\theta|s_0) = \frac{1}{2\pi} e^{-a^2} + \frac{a}{\sqrt{\pi}} \cos\theta \, e^{-a^2\sin^2\theta} \times \left(1 - Q(\sqrt{2a^2\cos^2\theta})\right). \quad (16)$$

The following can be easily verified.

$$f_{\theta|s_0}(\theta|s_0)|_{a=0} = \frac{1}{2\pi}, \qquad \left.\frac{df_{\theta|s_0}}{da}\right|_{a=0} = \frac{\cos\theta}{2\sqrt{\pi}},$$
$$\left.\frac{d^2 f_{\theta|s_0}}{da^2}\right|_{a=0} = \frac{\cos 2\theta}{\pi}, \qquad \left.\frac{df^3_{\theta|s_0}}{da^3}\right|_{a=0} = \frac{-3\cos\theta\sin^2\theta}{\sqrt{\pi}},$$
$$\left.\frac{df^4_{\theta|s_0}}{da^4}\right|_{a=0} = \frac{6\cos^2 2\theta}{\pi} - \frac{8\cos^4\theta}{\pi}.$$

Using the above derivatives, we can find the first through fourth derivatives of $P_{l,0}$ with respect to a at $a = 0$. Using the derivatives of $P_{l,0}$ and performing several algebraic operations, we arrive to the following Taylor expansion for $C_M(a)$ at $a = 0$:

$$C_M(a) = \phi_1(M)\, a^2 + \phi_2(M)\, a^3 + \phi_3(M)\, a^4 + o(a^4) \tag{17}$$
$$= \phi_1(M)\text{SNR} + \phi_2(M)\text{SNR}^{3/2} + \phi_3(M)\text{SNR}^2 + o(\text{SNR}^2) \tag{18}$$

where (18) follows due to the fact that $a = \sqrt{\text{SNR}}$. In the above expansion,

$$\phi_1(M) = \frac{M}{2\pi}\sin^2\frac{\pi}{M}\sum_{i=1}^{M}\cos^2\frac{2\pi i}{M}, \tag{19}$$

$$\phi_2(M) = \frac{M}{\pi\sqrt{\pi}}\left(\sin\frac{\pi}{M}\sin\frac{2\pi}{M} - \frac{M}{6}\sin^3\frac{\pi}{M}\right)\sum_{i=1}^{M}\cos^3\frac{2\pi i}{M}, \tag{20}$$

and

$$\phi_3(M) = -\frac{M^2}{16\pi}\sin^2\frac{2\pi}{M} + \frac{M(\pi+2)}{16\pi^2}\sin^2\frac{2\pi}{M}\sum_{i=1}^{M}\cos^2\frac{4\pi i}{M}$$
$$+ \left(\left(\frac{M^3}{12\pi^2} - \frac{M}{3\pi}\right)\sin^4\frac{\pi}{M} - \frac{M^2}{2\pi^2}\sin^2\frac{\pi}{M}\sin\frac{2\pi}{M}\right)\sum_{i=1}^{M}\cos^4\frac{2\pi i}{M}$$
$$+ \frac{M^2}{4\pi^2}\sin^2\frac{\pi}{M}\sin\frac{2\pi}{M}\sum_{i=1}^{M}\cos^2\frac{2\pi i}{M}. \tag{21}$$

We immediately conclude from (18) that $\dot{C}_M(0) = \phi_1(M)$. Note that the expansion includes the term $\text{SNR}^{3/2}$ which implies that $\ddot{C}_M(0) = \pm\infty$ for all M. However, it can be easily seen that $\phi_2(M) = 0$ for all $M \neq 3$, and at $M = 3$, $\phi_2(3) = 0.1718$. Therefore, while $\ddot{C}_3(0) = \infty$, $\ddot{C}_M(0) = 2\phi_3(M)$ for $M \neq 3$. Further algebraic steps and simplification yield (13) and (14). □

Remark: We should note that the first derivative expression (13) has previously been given in [1] through the bit energy expressions. In addition, Verdú in [3] has provided the second derivative expression for the special case of $M = 4$. Hence, the main novelty in Theorem 1 is the second derivative expression for general M. First and second derivative expressions are given together for completeness.

The following corollary provides the asymptotic behavior as $M \to \infty$.

Corollary 1: In the limit as $M \to \infty$, the first and second derivatives of the capacity at zero SNR converge to

$$\lim_{M\to\infty}\dot{C}_M(0) = \frac{\pi}{4} \text{ and } \lim_{M\to\infty}\ddot{C}_M(0) = \frac{\pi^2 - 8\pi + 8}{16}. \tag{22}$$

In the low-power regime, the tradeoff between bit energy and spectral efficiency is a key measure of performance. The normalized energy per bit can be obtained from

$$\frac{E_b}{N_0} = \frac{\text{SNR}}{C(\text{SNR})} \tag{23}$$

where $C(\text{SNR})$ is the channel capacity in bits/symbol. The maximum achievable spectral efficiency in bits/s/Hz is given by

$$\mathsf{C}\left(\frac{E_b}{N_0}\right) = C(\text{SNR}) \text{ bits/s/Hz} \tag{24}$$

if we, without loss of generality, assume that one symbol occupies a 1s × 1Hz time-frequency slot. Two important notions regarding the spectral-efficiency/bit-energy tradeoff in the low power regime are the bit-energy required at zero spectral efficiency,

$$\left.\frac{E_b}{N_0}\right|_{\mathsf{C}=0} = \frac{\log_e 2}{\dot{C}(0)}, \tag{25}$$

and the wideband slope,

$$S_0 = \frac{2(\dot{C}(0))^2}{-\ddot{C}(0)}, \tag{26}$$

which gives the slope of the spectral efficiency curve $\mathsf{C}(E_b/N_0)$ at zero spectral efficiency [3]. Therefore, $\left.\frac{E_b}{N_0}\right|_{\mathsf{C}=0}$ and S_0 constitute a linear approximation to the spectral efficiency curve in the low-SNR regime. Since these quantities depend only $\dot{C}(0)$ and $\ddot{C}(0)$, the bit energy at zero spectral efficiency and wideband slope achieved by M-ary PSK signals with a hard-decision detector can be readily obtained by using the formulas (13) and (14).

Corollary 2: The bit energy at zero spectral efficiency and wideband slope achieved by M-ary PSK signaling are given by

$$\left.\frac{E_b}{N_0}\right|_{\mathsf{C}=0} = \begin{cases} \frac{\pi}{2}\log_e 2 & M = 2 \\ \frac{4\pi}{M^2\sin^2\frac{\pi}{M}}\log_e 2 & M \geq 3 \end{cases} \tag{27}$$

and

$$S_0 = \begin{cases} \frac{3}{\pi - 1} & M = 2 \\ 0 & M = 3 \\ \frac{6}{\pi - 1} & M = 4 \\ \frac{\frac{M^4}{8\pi^2}\sin^4\frac{\pi}{M}}{-\psi(M)} & M \geq 5 \end{cases} \tag{28}$$

respectively.

As it will be evident in numerical results, generally the $\left.\frac{E_b}{N_0}\right|_{\mathsf{C}=0}$ is the minimum bit energy required for reliable transmission when $M \neq 3$. However, for $M = 3$, the minimum bit energy is achieved at a nonzero spectral efficiency.

Corollary 3: For 3-PSK modulation, the minimum bit energy is achieved at a nonzero spectral efficiency.

This corollary follows immediately from the fact that $\ddot{C}_3(0) = \infty$ which implies that the slope at zero SNR of $\text{SNR}/C_3(\text{SNR})$ is $-\infty$. This lets us conclude that the bit energy required at zero spectral efficiency cannot be the minimum one. The fact that 3-PSK achieves its minimum bit energy at a nonzero spectral efficiency is also pointed out in [2] through numerical results. Here, this result is shown analytically through the second derivative expression.

Figure 1 plots the spectral efficiency curves as a function of the bit energy for hard-detected PSK with different constellation sizes. As observed in this figure, the information-theoretic analysis conducted in this paper provides several practical design guidelines. We note that although 2-PSK and 4-PSK achieve the same minimum bit energy of 0.369 dB at zero spectral efficiency, 4-PSK is more efficient at low but nonzero spectral efficiency values due to its wideband slope being twice that of 2-PSK. 3-PSK is better than 2-PSK for

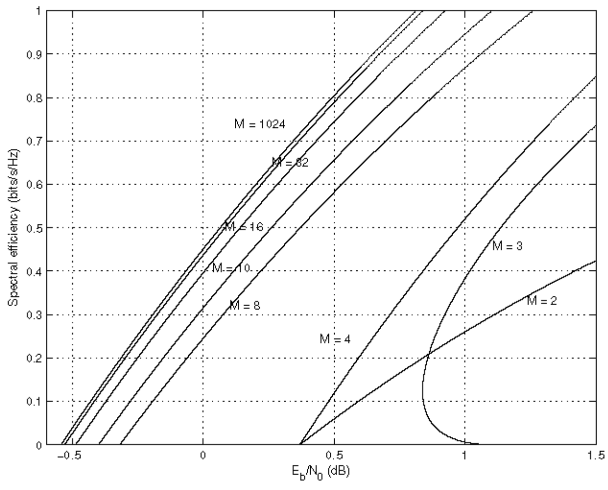

Fig. 1. Spectral efficiency $C(E_b/N_0)$ vs. bit energy E_b/N_0 for M-ary PSK with a hard-decision detection in the AWGN channel.

spectral efficiency values greater than 0.208 bits/s/Hz below which 3-PSK performes worse than both 2 and 4-PSK. 3-PSK achieves its minimum bit energy of 0.8383 dB at 0.124 bits/s/Hz. Operation below this level of spectral efficiency should be avoided as it only increases the energy requirements. We further observe that increasing the constellation size to 8 provides much improvement over 4-PSK. 8-PSK achieves a minimum bit energy of -0.318 dB. Further increase in M provides diminishing returns. For instance, there is little to be gained by increasing the constellation size more than 32 as 32-PSK achieves a minimum bit energy of -0.528 dB and the minimum bit energy as $M \to \infty$ is -0.542 dB. Note that -0.542 dB still presents a loss of approximately 1.05 dB with respect to the fundamental limit of -1.59 dB achieved by soft detection. We find that the wideband slopes of M = 8,10,16,32, and 1024 are 2.44, 2.53, 2.64, 2.69, and 2.71. The similarity of the wideband slope values is also apparent in the figure. As $M \to \infty$, the wideband slope is 2.717. Finally, note that the wideband slope of 3-PSK, as predicted, is 0.

IV. PSK over Fading Channels

A. Coherent Fading Channels

In this section, we consider fading channels and assume that the fading coefficients $\{h_k\}$ are known at the receiver but not at the transmitter. The only requirements on the fading coefficients are that their variations are ergodic and they have finite second moments. Due to the presence of receiver channel side information (CSI), scaled nearest point detection is employed, and the analysis follows along lines similar to those in the previous section. Hence, the treatment will be brief.

Note that in this case, the average capacity is

$$C_M(\text{SNR}) = \log M + \sum_{l=0}^{M-1} E_h\{P_{l,0,h} \log P_{l,0,h}\} \qquad (29)$$

where

$$P_{l,0,h} = \int_{\frac{(2l-1)\pi}{M}}^{\frac{(2l+1)\pi}{M}} f_{\theta|s_0,h}(\theta|s_0,h)\, d\theta \qquad (30)$$

and

$$f_{\theta|s_0,h}(\theta|s_0,h) = \frac{1}{2\pi} e^{-|h|^2\text{SNR}} + \sqrt{\frac{|h|^2\text{SNR}}{\pi}} \cos\theta\, e^{-|h|^2\text{SNR}\sin^2\theta} \times \left(1 - Q(\sqrt{2|h|^2\text{SNR}\cos^2\theta})\right) \qquad (31)$$

with $\text{SNR} = \mathcal{E}/N_0$. Through a similar analysis as in Section III, we have the following result on the derivatives of the capacity.

Theorem 2: The first and second derivatives of $C_M(\text{SNR})$ in nats per symbol at $\text{SNR} = 0$ are given by

$$\dot{C}_M(0) = \begin{cases} \frac{2}{\pi} E\{|h|^2\} & M = 2 \\ \frac{M^2}{4\pi} \sin^2 \frac{\pi}{M} E\{|h|^2\} & M \geq 3 \end{cases}, \qquad (32)$$

and

$$\ddot{C}_M(0) = \begin{cases} \frac{8}{3\pi}\left(\frac{1}{\pi} - 1\right) E\{|h|^4\} & M = 2 \\ \infty & M = 3 \\ \frac{4}{3\pi}\left(\frac{1}{\pi} - 1\right) E\{|h|^4\} & M = 4 \\ \psi(M) E\{|h|^4\} & M \geq 5 \end{cases} \qquad (33)$$

respectively, where $\psi(M)$ is given in (15).

Note that the first derivative and second derivatives of the capacity at zero SNR are essentially equal to the scaled versions of those obtained in the AWGN channel. The scale factors are $E\{|h|^2\}$ and $E\{|h|^4\}$ for the first and second derivatives, respectively.

In the fading case, we can define the received bit energy as

$$\frac{E_b^r}{N_0} = \frac{E\{|h|^2\}\text{SNR}}{C_M(\text{SNR})} \qquad (34)$$

as $E\{|h|^2\}\text{SNR}$ is the received signal-to-noise ratio. It immediately follows from Theorem 2 that $E_b^r/N_0|_{C=0}$ in the coherent fading channel is the same as that in the AWGN channel. On the other hand, the wideband slope is scaled by $(E\{|h|^2\})^2/E\{|h|^4\}$.

B. Noncoherent Fading Channels

In this section, we assume that neither the receiver nor the transmitter knows the fading coefficients $\{h_k\}$. We further assume that $\{h_k\}$ are i.i.d. proper complex Gaussian random variables with mean $E\{h_k\} = d \neq 0$ [3] and variance $E\{|h_k - d|^2\} = \gamma^2$. Now, the conditional probability density function of the channel output given the input is

$$f_{r|s_m}(r|s_m) = \frac{1}{\pi(\gamma^2|s_m|^2 + N_0)} e^{-\frac{|r - ds_m|^2}{\gamma^2|s_m|^2 + N_0}}. \qquad (35)$$

Recall that $\{s_m = \sqrt{\mathcal{E}} e^{j\theta_m}\}$ are the PSK signals and hence $|s_m| = \sqrt{\mathcal{E}}$ for all $m = 0, \ldots, M-1$. Due to this constant magnitude property, it can be easily shown that the maximum likelihood detector selects s_k as the transmitted signal if[4]

$$\text{Re}(rs_k^*) > \text{Re}(rs_i^*) \quad \forall i \neq k \qquad (36)$$

[3] $d \neq 0$ is required because phase cannot be used to transmit information in a noncoherent Rayleigh fading channel where $d = 0$.

[4] (36) is obtained when we assume, without loss of generality, that $d = |d|$.

where s_k^* is the complex conjugate of s_k, and Re denotes the operation that selects the real part. Therefore, the decision regions are the same as in the AWGN channel case.

In this case, the channel capacity is

$$C_M(\text{SNR}) = \log M + \sum_{l=0}^{M-1} P_{l,0} \log P_{l,0} \tag{37}$$

where

$$P_{l,0} = \int_{\frac{(2l-1)\pi}{M}}^{\frac{(2l+1)\pi}{M}} f_{\theta|s_0}(\theta|s_0)\, d\theta \tag{38}$$

and

$$f_{\theta|s_0}(\theta|s_0) = \frac{1}{2\pi} e^{-\frac{|d|^2\text{SNR}}{\gamma^2\text{SNR}+1}} \tag{39}$$

$$+\sqrt{\frac{|d|^2\text{SNR}}{\pi(\gamma^2\text{SNR}+1)}} \cos\theta\, e^{-\frac{|d|^2\text{SNR}}{\gamma^2\text{SNR}+1}\sin^2\theta}$$

$$\times\left(1 - Q\left(\sqrt{2\frac{|d|^2\text{SNR}}{\gamma^2\text{SNR}+1}\cos^2\theta}\right)\right). \tag{40}$$

The following results provide the first and second derivatives of the capacity at zero SNR, and the bit energy and wideband slope in the low-SNR regime.

Theorem 3: The first and second derivatives of $C_M(\text{SNR})$ in nats per symbol at $\text{SNR} = 0$ are given by

$$\dot{C}_M(0) = \begin{cases} \frac{2|d|^2}{\pi} & M = 2 \\ \frac{M^2|d|^2}{4\pi}\sin^2\frac{\pi}{M} & M \geq 3 \end{cases}, \tag{41}$$

and

$$\ddot{C}_M(0) = \begin{cases} \frac{8}{3\pi}\left(\frac{1}{\pi} - 1\right)|d|^4 - \frac{4|d|^2\gamma^2}{\pi} & M = 2 \\ \infty & M = 3 \\ \frac{4}{3\pi}\left(\frac{1}{\pi} - 1\right)|d|^4 - \frac{4|d|^2\gamma^2}{\pi} & M = 4 \\ \psi(M)|d|^4 - \frac{|d|^2\gamma^2}{2\pi}M^2\sin^2\frac{\pi}{M} & M \geq 5 \end{cases} \tag{42}$$

respectively, where $\psi(M)$ is given in (15).

Corollary 4: In the limit as $M \to \infty$, the first and second derivatives of the capacity at zero SNR converge to

$$\lim_{M\to\infty} \dot{C}_M(0) = \frac{\pi|d|^2}{4}. \tag{43}$$

and

$$\lim_{M\to\infty} \ddot{C}_M(0) = \frac{(\pi^2 - 8\pi + 8)|d|^4}{16} - \frac{|d|^2\gamma^2\pi}{2}. \tag{44}$$

In the noncoherent fading case, the received bit energy is

$$\frac{E_b^r}{N_0} = \frac{(|d|^2 + \gamma^2)\text{SNR}}{C_M(\text{SNR})}. \tag{45}$$

Corollary 5: The received bit energy at zero spectral efficiency and wideband slope achieved by M-ary PSK signaling are given by

$$\left.\frac{E_b}{N_0}\right|_{\mathsf{C}=0} = \begin{cases} \frac{\pi}{2}\left(1 + \frac{1}{\mathsf{K}}\right)\log_e 2 & M = 2 \\ \frac{4\pi}{M^2\sin^2\frac{\pi}{M}}\left(1 + \frac{1}{\mathsf{K}}\right)\log_e 2 & M \geq 3 \end{cases} \tag{46}$$

and

$$S_0 = \begin{cases} \frac{3}{\pi - 1 + \frac{3\pi}{2\mathsf{K}}} & M = 2 \\ 0 & M = 3 \\ \frac{6}{\pi - 1 + \frac{3\pi}{\mathsf{K}}} & M = 4 \\ \frac{\frac{M^4}{8\pi^2}\sin^4\frac{\pi}{M}}{-\psi(M) + \frac{1}{2\pi\mathsf{K}}M^2\sin^2\frac{\pi}{M}} & M \geq 5 \end{cases} \tag{47}$$

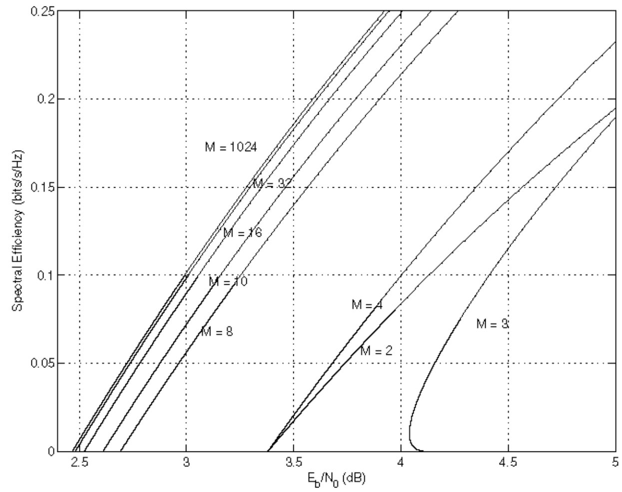

Fig. 2. Spectral efficiency $\mathsf{C}(E_b/N_0)$ vs. bit energy E_b/N_0 in the noncoherent Rician fading channel with Rician factor $\mathsf{K} = \frac{|d|^2}{\gamma^2} = 1$.

respectively, where $\psi(M)$ is given in (15), and $\mathsf{K} = \frac{|d|^2}{\gamma^2}$ is the Rician factor.

Remark: If we let $|d| = 1$ and $\gamma^2 = 0$, or equivalently let $\mathsf{K} \to \infty$, the results provided above coincide with those given for the AWGN channel.

Fig. 2 plots the spectral efficiency curves as a function of the bit energy for M-ary PSK transmission over the noncoherent Rician fading channel with $\mathsf{K} = 1$. Note that conclusions similar to those given for Fig. 1 also apply for Fig. 2. The main difference between the figures is the substantial increase in the bit energy values as a penalty of not knowing the channel. For instance, 2 and 4-PSK now achieves a minimum bit energy of 3.379 dB while 8-PSK attains 2.692 dB. As $M \to \infty$, the minimum bit energy goes to 2.467 dB.

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