

Noncoherent Rician Fading Channel—Part II: Spectral Efficiency in the Low-Power Regime

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Abstract—Transmission of information over a discrete-time memoryless Rician fading channel is considered, where neither the receiver nor the transmitter knows the fading coefficients. The spectral-efficiency/bit-energy tradeoff in the low-power regime is examined when the input has limited peakedness. It is shown that if a fourth-moment input constraint is imposed, or the input peak-to-average power ratio is limited, then in contrast to the behavior observed in average-power-limited channels, the minimum bit energy is not always achieved at zero spectral efficiency. The low-power performance is also characterized when there is a fixed peak limit that does not vary with the average power. A new signaling scheme that overlays phase-shift keying on ON-OFF keying (OOK) is proposed and shown to be optimally efficient in the low-power regime.

Index Terms—Fading channels, low-power regime, memoryless fading, peak constraints, Rician fading, spectral efficiency.

I. INTRODUCTION

EMERGING wireless systems, such as wideband code division multiple access (WCDMA) and impulse radio, operate at wide bandwidths. These systems can achieve higher data rates, are more immune to the deleterious effects of multipath fading, and require low power consumption. Many other wireless communication systems, such as satellite, deep-space, and sensor networks, also operate in the low-power regime, where both spectral efficiency (rate in bits per second divided by bandwidth in hertz) and energy per bit are low. For these systems, information-theoretic results on the spectral-efficiency/bit-energy tradeoff, which reflects the fundamental tradeoff between bandwidth and power, provide insightful results leading to the more efficient use of resources in the low signal-to-noise ratio (SNR) regime.

Verdú [4] has recently analyzed the spectral efficiency of a general class of average-power-limited channels characterizing the optimal bandwidth–power tradeoff in the wideband regime. In particular, it is shown in [4] that when the receiver has imperfect fading-side information, input signals

Manuscript received January 31, 2003; revised January 4, 2004 and June 17, 2004; accepted June 23, 2004. The editor coordinating the review of this paper and approving it for publication is G. Leus. This work was supported by the US Army Research Laboratory under Contract DAAD 19-01-2-0011. The material in this paper was presented in part at the 40th Annual Allerton Conference on Communication, Control, and Computing, Monticello, IL, October, 2002.

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Digital Object Identifier 10.1109/TWC.2005.853971

with increasingly higher peak power are required to achieve the capacity as $\text{SNR} \rightarrow 0$. On the other hand, limiting the peakedness of the input signals, when neither the receiver nor the transmitter knows the fading, is known to have a significant impact on the achievable spectral efficiency in the low-power regime [11]–[13]. In this paper, we continue our study of noncoherent Rician fading channels begun in Part I [1] and consider the minimum energy per bit required for reliable communication when the input signals have limited peakedness. The organization of the paper is as follows. In Section II, we review the basic measures of interest in the low-power regime proposed in [4]. In Section III, we analyze the spectral-efficiency/bit-energy tradeoff in the noncoherent Rician fading channel when, in addition to the average-power limitation, the input is subject to a fourth-moment or a peak-power constraint. Finally, in Section IV, we show efficient signaling schemes in the low-power regime, while Section V contains our conclusions.

II. PRELIMINARIES

In the low-power regime, the spectral-efficiency/bit-energy tradeoff is the key concept capturing the tradeoff between bandwidth and power. We will denote the spectral efficiency as a function of bit energy by $C(E_b/N_0)$. If we assume, without loss of generality, that one complex symbol occupies a $1\text{-s} \times 1\text{-Hz}$ time–frequency slot, then the maximum achievable spectral efficiency can be obtained from the Shannon capacity (bits per symbol)

$$C\left(\frac{E_b}{N_0}\right) = C(\text{SNR}) \text{ bits/s/Hz} \quad (1)$$

where

$$\frac{E_b}{N_0} = \frac{\text{SNR}}{C(\text{SNR})} \quad (2)$$

is the bit energy normalized to the noise power. However, the Shannon capacity, giving the full characterization of the spectral-efficiency/bit-energy function, is either not known or must be numerically computed for most fading channels whose realizations are not fully known at the receiver. Hence, one needs to resort to approximation methods to examine the spectral efficiency of a wide class of fading channels. In the low-SNR regime, first-order linear approximation of the spectral-efficiency function provides an excellent match, and involves only the slope of the spectral-efficiency curve and the

bit energy at zero spectral efficiency. The bit energy at zero spectral efficiency, which depends only on the first derivative of the capacity at zero SNR, i.e.

$$\left. \frac{E_b}{N_0} \right|_{C=0} = \lim_{\text{SNR} \rightarrow 0} \frac{\text{SNR}}{C(\text{SNR})} = \frac{\log_e 2}{\dot{C}(0)} \quad (3)$$

is a relevant measure only in the asymptotic regime of infinite bandwidth. Verdú [4] has recently given the following formula for the wideband slope defined as the slope of the spectral-efficiency curve $C(E_b/N_0)$ in bits per second per hertz per 3 dB at zero spectral efficiency:

$$S_0 \stackrel{\text{def}}{=} \lim_{\substack{E_b/N_0 \uparrow \\ E_b/N_0 \downarrow}} \left. \frac{C\left(\frac{E_b}{N_0}\right)}{10 \log_{10} \frac{E_b}{N_0} - 10 \log_{10} \frac{E_b}{N_0}} \right|_{C=0} = \frac{2 \left(\dot{C}(0) \right)^2}{-\ddot{C}(0)} \quad (4)$$

where $\dot{C}(0)$ and $\ddot{C}(0)$ denote the first and second derivatives of the capacity at zero SNR in nats. The wideband slope closely approximates the growth of the spectral-efficiency curve in the low-power regime and hence, is proposed as a new tool providing insightful results when bandwidth is a resource to be conserved.

For average-power-limited channels, the bit energy required for reliable communications decreases monotonically with decreasing spectral efficiency, and the minimum bit energy is achieved at zero spectral efficiency, $E_b/N_{0,\min} = E_b/N_0|_{C=0}$. Hence, for fixed-rate transmission, reduction in the required power comes only at the expense of increased bandwidth. By assuming an ergodic fading process of finite second moment, Lapidoth and Shamai [5] have shown that the minimum received bit energy for an average-power-limited discrete-time single-input single-output fading channel with Gaussian noise is -1.59 dB. Recently, Verdú [4] has independently proven that the minimum received bit energy of -1.59 dB is achieved in a general class of average-power-limited multiple-input multiple-output fading channels as long as the additive background noise is Gaussian. This result holds regardless of the availability of the fading knowledge at the receiver and transmitter. On the other hand, it is shown in [4] that having imperfect receiver-side information has a tremendous effect on the wideband slope. Although there is a positive slope when there is perfect receiver-channel-side information, imperfect fading knowledge at the receiver results in zero wideband slope, and in this case, flash signaling is required to achieve the capacity in the low-power regime. Hence, achieving the minimum bit energy becomes very demanding in terms of both bandwidth and the peak-to-average ratio of the transmitted signal.

III. SPECTRAL EFFICIENCY VERSUS BIT ENERGY

In this section, we consider the noncoherent Rician fading channel model

$$y_i = mx_i + a_i x_i + n_i \quad (5)$$

studied in Part I of this paper [1] and investigate the spectral efficiency versus energy-per-information-bit tradeoff in the low-power regime when the peakedness of the input signals is limited by a fourth-moment or a peak-power constraint. Moreover, at the end of this section, we comment on the spectral efficiency in the low-SNR regime of the average-power-limited Rician channel with phase noise.

A. Second- and Fourth-Moment-Limited Input

We first consider the case in which the input is subject to the following second- and fourth-moment amplitude constraints:

$$E\{|x_i|^2\} \leq P_{\text{av}} \quad \forall i \quad (6)$$

$$E\{|x_i|^4\} \leq \kappa P_{\text{av}}^2 \quad \forall i \quad (7)$$

where $1 < \kappa < \infty$. We have seen in the previous section that the first-order linear approximation of the spectral-efficiency/bit-energy function in the low-power regime is determined by the first and second derivatives of the capacity at zero SNR. Under quite general conditions on the input and the channel, Prelov and Verdú [6] obtained the exact asymptotic second-order behavior of the mutual information between the channel input and the output for vanishing SNR

$$I(\mathbf{x}, \mathbf{H}\mathbf{x} + \mathbf{n}) = E\left\{ \left\| \bar{\mathbf{H}}(\mathbf{x} - E[\mathbf{x}]) \right\|^2 \right\} \frac{\log e}{N_0} + E\left\{ \text{trace}(\text{cov}^2(\mathbf{H}\mathbf{x}|\mathbf{x})) \right\} \frac{\log e}{2N_0^2} - \text{trace}(\text{cov}^2(\mathbf{H}\mathbf{x})) \frac{\log e}{2N_0^2} + o(N_0^{-2}) \quad (8)$$

where \mathbf{H} is an $m \times n$ complex matrix of random fading coefficients satisfying $E\{\|\mathbf{H}\|^{4+\alpha}\} < \infty$ for some $\alpha > 0$, $\bar{\mathbf{H}} = E\{\mathbf{H}\}$, N_0 is the one-sided noise spectral level, and $o(x)/x \rightarrow 0$ as $x \rightarrow 0$. The only assumption on the input is that its probability distribution satisfies

$$P(\|\mathbf{x}\| > \delta) \leq \exp\{-\delta^v\} \quad (9)$$

for all sufficiently large $\delta > 0$, where $v > 0$ is a positive constant. Using the above result, we can obtain the first and second derivatives of the mutual information at zero SNR for the noncoherent Rician fading channel (5) with second- and fourth-moment input constraints.

Proposition 1: For the Rician channel model (5), with the Rician factor $K = |m|^2/\gamma^2 > 0$ and input constraints (6) and (7), the first and second derivatives of the capacity at SNR = 0 are

$$\dot{C}(0) = |m|^2 \text{ and } \ddot{C}(0) = \kappa\gamma^4 - (|m|^2 + \gamma^2)^2 \quad (10)$$

respectively.

Proof: For the noncoherent Rician fading channel model, we have established in [1] that the capacity-achieving input distribution has finite support. Hence, the optimal input distribution satisfies the condition (9). Therefore, the proposition

follows easily by specializing the general result (8) of Prelov and Verdú to the Rician channel model (5), i.e.

$$I(x, mx + ax + n) = |m|^2 \frac{E\{|x|^2\}}{N_0} + \frac{1}{2} \gamma^4 \frac{E\{|x|^4\}}{N_0^2} - \frac{1}{2} (|m|^2 + \gamma^2)^2 \left(\frac{E\{|x|^2\}}{N_0} \right)^2 + o(N_0^{-2}). \quad (11)$$

Since, by our assumption, $|m| > 0$, the first term on the right-hand side of (11) is maximized by having $E\{|x|^2\} = P_{av}$. Note that since the other terms characterize the second- and higher order behavior, the average-power constraint should be satisfied with equality to achieve the first derivative of the capacity at zero SNR. Notice also that the second term is maximized by having $E\{|x|^4\} = \kappa P_{av}^2$, and hence, the asymptotic capacity expression becomes

$$C = |m|^2 \left(\frac{P_{av}}{N_0} \right) + \frac{1}{2} \left(\kappa \gamma^4 - (|m|^2 + \gamma^2)^2 \right) \left(\frac{P_{av}}{N_0} \right)^2 + o(N_0^{-2}) = |m|^2 \text{SNR} + \frac{1}{2} \left(\kappa \gamma^4 - (|m|^2 + \gamma^2)^2 \right) \text{SNR}^2 + o(\text{SNR}^2) \quad (12)$$

from which the result follows. \blacksquare

Remark 1: It can be easily seen from the asymptotic expression (11) that for the Rayleigh channel ($K = 0$) with input constraints (6) and (7), $\dot{C}(0) = 0$. We also note that Rao and Hassibi [16] have recently obtained the second-order asymptotic expression of the mutual information in the multiple-antenna Rayleigh block fading channel when the fourth-order moment of the input is finite, and shown that the mutual information is zero to first order in SNR.

Remark 2: Note that the first derivative depends only on the strength of the line-of-sight component $|m|^2$. Hence, for fixed $|m|$, capacity curves under fourth-moment constraints with different but finite κ values have the same slope at zero SNR. On the other hand, the second derivative depends on both $|m|^2$ and κ and is positive if $\kappa > (1 + K)^2$ where $K = |m|^2/\gamma^2$ is the Rician factor. Therefore, unlike the average-power-limited channels, where the capacity is a concave function of the SNR, the capacity curve in this case is a convex function locally around $\text{SNR} = 0$. Finally, as a comparison, if there is no fourth-moment constraint (i.e., $\kappa = \infty$), it is shown in [4] that $\dot{C}(0) = |m|^2 + \gamma^2$, $\ddot{C}(0) = -\infty$.

The above proposition reveals an interesting property. According to [19]

$$\dot{C}(0) = |m|^2 = N_0 \lim_{|x_0| \rightarrow 0} \frac{D(f_{y|x=x_0} \| f_{y|x=0})}{|x_0|^2} \quad (13)$$

$$= N_0 \lim_{|x_0| \rightarrow 0} \frac{\frac{|m|^2 + \gamma^2}{N_0} |x_0|^2 - \log_e \left(\frac{\gamma^2}{N_0} |x_0|^2 + 1 \right)}{|x_0|^2} \quad (14)$$

which shows that any ON-OFF signaling scheme that satisfies the second- and fourth-moment input constraints and whose ON level approaches the origin as $\text{SNR} \rightarrow 0$ achieves the first derivative of the capacity.

Moreover, imposing a fourth-moment or a peak-power constraint is essentially the same as far as the first derivative of the capacity is considered. In [14], it is shown that for a general class of memoryless channels with average- and peak-power constraints, $E\{|x|^2\} \leq P_{av}$, $|x|^2 \leq \kappa P_{av}$ where $\kappa < \infty$, the capacity has the following asymptotic expression as $\text{SNR} \rightarrow 0$:

$$C(\text{SNR}) = \frac{1}{2} N_0 \Lambda \text{SNR} + o(\text{SNR}) \quad (15)$$

where Λ is the largest eigenvalue of the Fisher information matrix [21]. We note that the limiting expression on the right-hand side of (13) is equal to one-half the largest eigenvalue of the Fisher information matrix which, in our case, is $K = E_{f_{y|x=0}}\{\mathbf{v}(y, 0) \mathbf{v}(y, 0)^T\} = (2|m|^2/N_0)\mathbf{I}$ where

$$\mathbf{v}(y, x) = \begin{bmatrix} \frac{\partial \log f_{y|x}}{\partial x_r} \\ \frac{\partial \log f_{y|x}}{\partial x_i} \end{bmatrix} \quad (16)$$

x_r and x_i denote the real and imaginary parts of x , respectively, and \mathbf{I} is the 2×2 identity matrix.

For the average-power-limited Rician fading channel (without fourth-moment constraint), the following derivative can be obtained: $\dot{C}(0) = |m|^2 + \gamma^2 = N_0 \lim_{|x_0| \rightarrow \infty} [D(f_{y|x=x_0} \| f_{y|x=0})/|x_0|^2]$, where in this case, the ON level should escape to infinity to achieve the first derivative of the capacity.

Having obtained analytical expressions for the first and second derivatives of the capacity at zero SNR, we now find the bit energy required at zero spectral efficiency and the wideband slope. We first note that the normalized received bit energy in the Rician channel has the following formula

$$\frac{E_b^r}{N_0} = \frac{E\{|m + a|^2\} \text{SNR}}{C(\text{SNR})} = \frac{(|m|^2 + \gamma^2) \text{SNR}}{C(\text{SNR})}. \quad (17)$$

Corollary 1: For the Rician fading channel (5), subject to input constraints (6) and (7), the normalized received bit energy E_b^r/N_0 required at zero spectral efficiency and the wideband slope are

$$\frac{E_b^r}{N_0} \Big|_{\text{C}=0} = \left(1 + \frac{1}{K} \right) \log_e 2 \text{ and } S_0 = \frac{2K^2}{(1 + K)^2 - \kappa} \quad (18)$$

respectively, where $K = |m|^2/\gamma^2$ is the Rician factor.

Proof: The received bit energy required at zero spectral efficiency is obtained by letting $\text{SNR} \rightarrow 0$:

$$\begin{aligned} \frac{E_b^r}{N_0} \Big|_{\text{C}=0} &= \lim_{\text{SNR} \rightarrow 0} \frac{(|m|^2 + \gamma^2) \text{SNR}}{C(\text{SNR})} \\ &= \frac{(|m|^2 + \gamma^2) \log_e 2}{\dot{C}(0)} \\ &= \left(1 + \frac{\gamma^2}{|m|^2} \right) \log_e 2. \end{aligned}$$

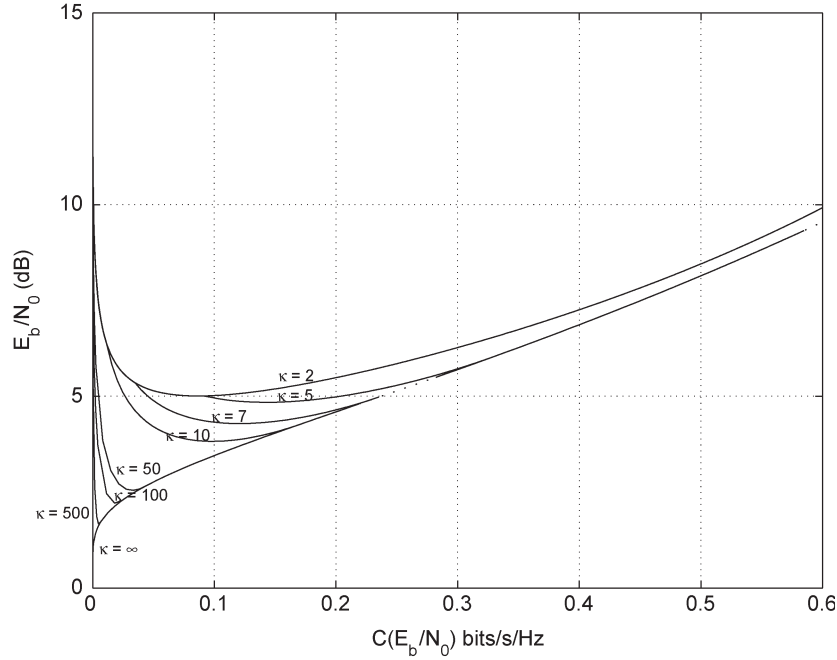


Fig. 1. E_b/N_0 (dB) versus spectral efficiency $C(E_b/N_0)$ (bits per second per hertz) for the Rayleigh channel. $K = 0$.

The wideband slope expression is obtained by inserting the first- and second-derivative expressions in (10) into (4). Moreover, for the Rayleigh channel where $\dot{C}(0) = 0$, it can be easily seen that $E_b^r/N_0|_{C=0} = \infty$ and $S_0 = 0$.

Remark 3: As long as a fourth-moment constraint is imposed, the bit energy required at zero spectral efficiency (or equivalently at infinite bandwidth) depends only on the Rician factor K , and

$$\begin{aligned} \frac{E_b^r}{N_0} \Big|_{C=0} &\rightarrow \infty && \text{as } K \rightarrow 0 \\ \frac{E_b^r}{N_0} \Big|_{C=0} &\rightarrow -1.59 \text{ dB} && \text{as } K \rightarrow \infty \end{aligned} \quad (19)$$

which also appeals intuitively because as K increases, the channel becomes more Gaussian, and for the unfaded Gaussian channel $E_b^r/N_0|_{C=0} = \log_e 2 = -1.59$ dB. For the Rayleigh fading channel, i.e., $|m| = 0$, the bit energy required at zero spectral efficiency is infinite. Therefore, reliable communications is not possible at this point. This is in stark contrast with the behavior observed in average-power-limited channels where the bit energy required at zero spectral efficiency is indeed the minimum one. On the other hand, for the Rician fading channel where $|m| > 0$, the required bit energy is finite.

Remark 4: For average-power-limited channels, the wideband slope is always nonnegative. In the noncoherent Rician fading channel subject to second- and fourth-moment input limitations, we again observe a markedly different behavior. From (18), we see that if $\kappa > (1 + K)^2$, then the wideband slope is negative, leading to the conclusion that the minimum bit energy is achieved at a nonzero spectral efficiency $C^* > 0$. In this case, as observed in [4, p. 1341], one should avoid operating in the region where the spectral efficiency is lower than C^* because decreasing the spectral efficiency further in this region

(i.e., increasing the bandwidth for fixed-rate transmission) only increases the required power.

The following bounds on the minimum received bit energy are easily obtained: $\log_e 2 \leq E_b^r/N_{0\min} \leq [1 + (1/K)] \log_e 2$, where the lower bound is the minimum received bit energy when there is only an average-power constraint, and the upper bound is the received bit energy required at zero spectral efficiency when the input is subject to a fourth-moment constraint. Note that the upper bound is loose in the Rayleigh case where $K = 0$. However, the larger the Rician factor K , the smaller the gap between the upper and lower bounds. If $\kappa \leq (1 + K)^2$, then the wideband slope is positive and, based on numerical evidence, we conjecture that for a large-enough value of K , the minimum bit energy is achieved at zero spectral efficiency and is equal to the bit-energy expression in (18).

Figs. 1 and 2 plot the E_b^r/N_0 (decibel) versus $C(E_b^r/N_0)$ (bits per second per hertz) curves for the Rayleigh and Rician ($K = 1$) channels, respectively, for various values of κ . In the Rayleigh fading channel, for any finite κ , the bit-energy curve is bowl-shaped, achieving its minimum at a nonzero spectral efficiency C^* . Therefore, for any $E_b/N_0 > E_b/N_{0\min}$, there are two spectral efficiencies $C_1 < C_2$ such that $E_b/N_0 = E_b/N_0(C_1) = E_b/N_0(C_2)$. In this case, one should avoid the low-power regime and operate at C_2 where, for the same power and rate, less bandwidth is required. In the Rician fading channel ($K = 1$), we observe the same behavior when $\kappa > (1 + K)^2 = 4$. The minimum bit energy is achieved at a nonzero spectral efficiency. However, note that now the bit energy required at zero spectral efficiency is finite and is the same for all finite κ . If $\kappa \leq 4$, the bit energy decreases monotonically with decreasing spectral efficiency and the minimum bit energy is achieved at zero spectral efficiency. Therefore, in this case, the bandwidth–power tradeoff is the usual one that we encounter in average-power-limited channels; i.e., for

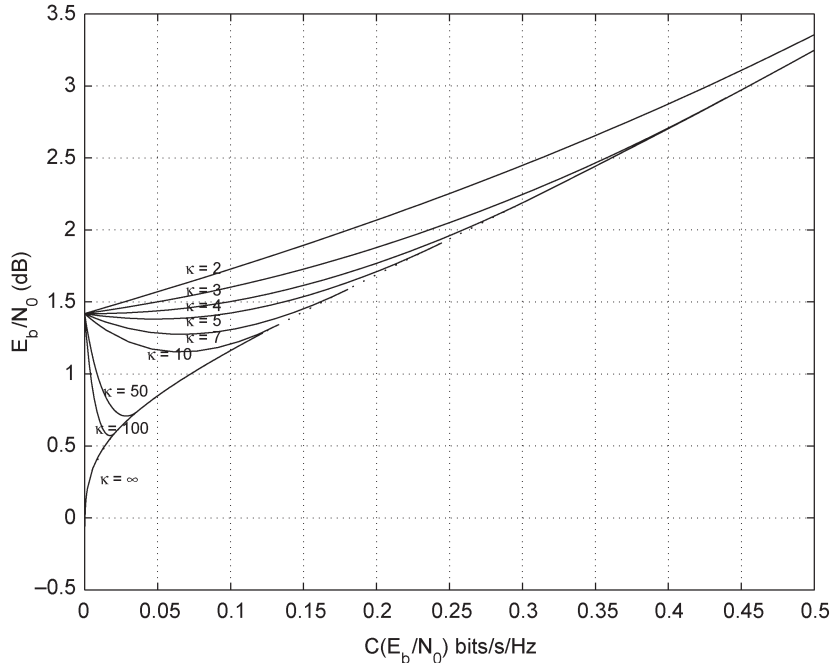


Fig. 2. E_b/N_0 (dB) versus spectral efficiency $C(E_b/N_0)$ (bits per second per hertz) for the Rician channel with $K = 1$.

fixed-rate transmission, increasing the bandwidth decreases the power required for reliable communications. Similar conclusions are drawn by observing Fig. 3, where the Rician factor has increased to $K = 4$. We note that all the minimum bit-energy points other than that attained in the Rayleigh channel with $\kappa = 2$ are achieved by a two-mass-point distribution in the following form

$$F(|x|) = \left(1 - \frac{1}{\kappa}\right) u(|x|) + \frac{1}{\kappa} u\left(|x| - \sqrt{\kappa N_0 \text{SNR}}\right). \quad (20)$$

An interesting observation is that in the Rayleigh channel, for sufficiently low SNR values, the optimal input satisfies $E\{|x|^2\} < P_{\text{av}}$ and $E\{|x|^4\} = \kappa P_{\text{av}}^2$, and hence, has a kurtosis higher than κ . In the case where $\kappa = 2$, a two-mass-point distribution with $E\{|x|^2\} < P_{\text{av}}$ achieves the minimum bit energy. Note that for the Rayleigh channel, the second-order asymptotic term in (11) is increased by decreasing the second moment while satisfying the fourth-moment constraint with equality.

B. Average- and Peak-Power-Limited Input

In this section, we impose a peak-power constraint, which is a more stringent approach than constraining the fourth moment of the amplitude. We analyze two cases: limited peak-to-average power ratio and limited peak power. In contrast to the first case, no constraint on the peak-to-average ratio is imposed in the second case where the input is subject to a fixed peak-power limit that does not vary with the average-power constraint.

1) *Limited Peak-to-Average Power Ratio:* We first consider the case in which the transmitter peak-to-average power ratio is

limited, and hence, the input, in addition to the average-power constraint (6), is subject to

$$|x_i|^2 \stackrel{\text{a.s.}}{\leq} \kappa P_{\text{av}} \quad \forall i \quad (21)$$

where $1 \leq \kappa < \infty$. The following result characterizes the spectral-efficiency/bit-energy tradeoff in the low-power regime.

Proposition 2: For the Rician fading channel (5), with average- and peak-power limitations (6) and (21), respectively, the normalized received bit energy E_b^r/N_0 required at zero spectral efficiency and the wideband slope are

$$\left. \frac{E_b^r}{N_0} \right|_{C=0} = \left(1 + \frac{1}{K}\right) \log_e 2 \text{ and } S_0 = \frac{2K^2}{(1+K)^2 - \kappa} \quad (22)$$

respectively, where $K = |m|^2/\gamma^2$ is the Rician factor.

Proof: Note that since the input is subject to a peak constraint (21), the condition (9) is satisfied, and hence, the input–output mutual information has again the same asymptotic expression (11). It is easily observed from (11) that in the Rayleigh channel where $|m| = 0$, $\dot{C}(0) = 0$. If $K > 0$, then to achieve the first derivative of the capacity at zero SNR, we must have $E\{|x|^2\} = P_{\text{av}}$. The following lemma gives the maximum value of the fourth moment of the amplitude when the input is subject to (6) and (21).

Lemma 1: Consider a nonnegative real random variable $|x|$. Then we have

$$\sup_{\substack{F_{|x|} \\ E\{|x|^2\} \leq P_{\text{av}} \\ |x|^2 \leq \kappa P_{\text{av}} \text{ a.s.}}} E\{|x|^4\} = \kappa P_{\text{av}}^2. \quad (23)$$

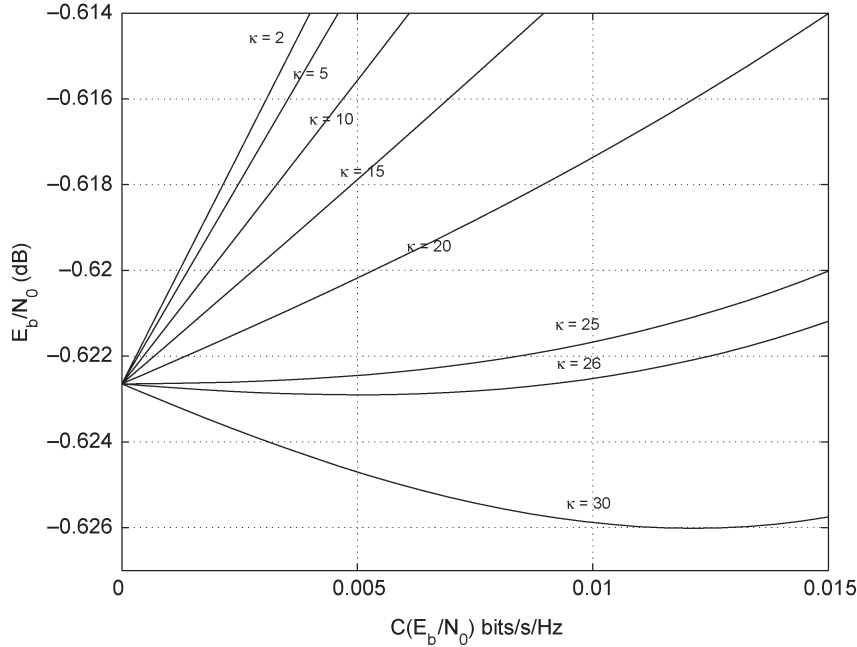


Fig. 3. E_b/N_0 (dB) versus spectral efficiency $C(E_b/N_0)$ (bits per second per hertz) for the Rician channel with $K = 4$.

Furthermore, the two-mass-point discrete distribution (24) achieves this supremum.

$$F_0^*(|x|) = \left(1 - \frac{1}{\kappa}\right) u(|x|) + \frac{1}{\kappa} u\left(|x| - \sqrt{\kappa P_{av}}\right). \quad (24)$$

Proof: Following the approach in [2] to find the Kuhn-Tucker condition, it is easily established that a sufficient and necessary condition for the distribution F_0 to achieve the supremum in (23) is that there exists $\lambda \geq 0$ such that

$$|x|^4 - \lambda|x|^2 \leq M - \lambda P_{av} \quad \forall |x| \in [0, \sqrt{\kappa P_{av}}] \quad (25)$$

$$= M - \lambda P_{av} \quad \forall |x| \in E_0 \quad (26)$$

where E_0 is the set of points of increase of F_0 and M is the supremum value. The two-mass-point distribution F_0^* defined in (24) satisfies these constraints and achieves $M = \kappa P_{av}^2$.

By the above Lemma and the fact that the average-power constraint has to be satisfied with equality to achieve the first derivative, the asymptotic capacity expression for the Rician channel with $K > 0$ becomes

$$C(\text{SNR}) = |m|^2 \text{SNR} + \frac{1}{2} \left(\kappa \gamma^4 - (|m|^2 + \gamma^2)^2 \right) \text{SNR}^2 + o(\text{SNR}^2) \quad (27)$$

from which we see that $\dot{C}(0) = |m|^2$ and $\ddot{C}(0) = \kappa \gamma^4 - (|m|^2 + \gamma^2)^2$. Then, the result in (22) is easily obtained from these derivative expressions similarly as in the proof of Corollary 1. ■

Remark 5: Note that with average- and peak-power constraints, we obtain the same bit-energy and wideband-slope expressions as in (18), where the input is subject to $E\{|x|^2\} \leq P_{av}$ and $E\{|x|^4\} \leq \kappa P_{av}^2$. Thus, we conclude that imposing a fourth-moment (7) or a peak constraint in the form (21) has the same effect in the low-power regime.

2) *Limited Peak Power:* In this section, we assume that the transmitter is limited in peak power and there is no constraint on the peak-to-average power ratio. Hence, the input, in addition to the average-power constraint (6), is subject to

$$|x_i|^2 \stackrel{\text{a.s.}}{\leq} \nu \quad \forall i \quad (28)$$

where ν is a fixed peak limit that does not vary with the average-power constraint P_{av} . Notice in this case that as $P_{av} \downarrow 0$, the peak-to-average ratio increases without bound. Recently, Sethuraman and Hajek [15] analyzed the capacity per unit energy of Gaussian fading channels with memory under similar average- and peak-power constraints. Considering the memoryless Rician fading channel, we obtain the following result on the minimum bit energy and wideband slope.

Proposition 3: For the Rician fading channel (5), with input constraints (6) and (28), and fixed noise density N_0 ,¹ the minimum received bit energy and wideband slope are

$$\frac{E_b^r}{N_0 \min} = \frac{\log_e 2}{1 - \frac{1}{K+1} \frac{\log_e(1+\eta)}{\eta}} \text{ and } S_0 = \begin{cases} \frac{2(\eta(K+1) - \log_e(1+\eta))^2}{-1 + \frac{1}{1-\eta^2} \exp\left(\frac{2K\eta^2}{1-\eta^2}\right) I_0\left(\frac{2K\eta}{1-\eta^2}\right)}, & \eta < 1 \\ 0, & \eta \geq 1 \end{cases} \quad (29)$$

respectively, where $K = |m|^2/\gamma^2$ is the Rician factor, $\eta = (\gamma^2/N_0)\nu$ is the normalized peak-power limit, and I_0 is the zeroth-order modified Bessel function of the first kind. Moreover, input signaling that satisfies

$$E\{|x_{\text{SNR}}|^2\} = P_{av} = N_0 \text{SNR}, \quad |x_{\text{SNR}}|^2 \stackrel{\text{a.s.}}{\leq} \nu \quad (30)$$

¹ N_0 is fixed so that SNR varies only with P_{av} , and the peak SNR constraint is kept constant at ν/N_0 .

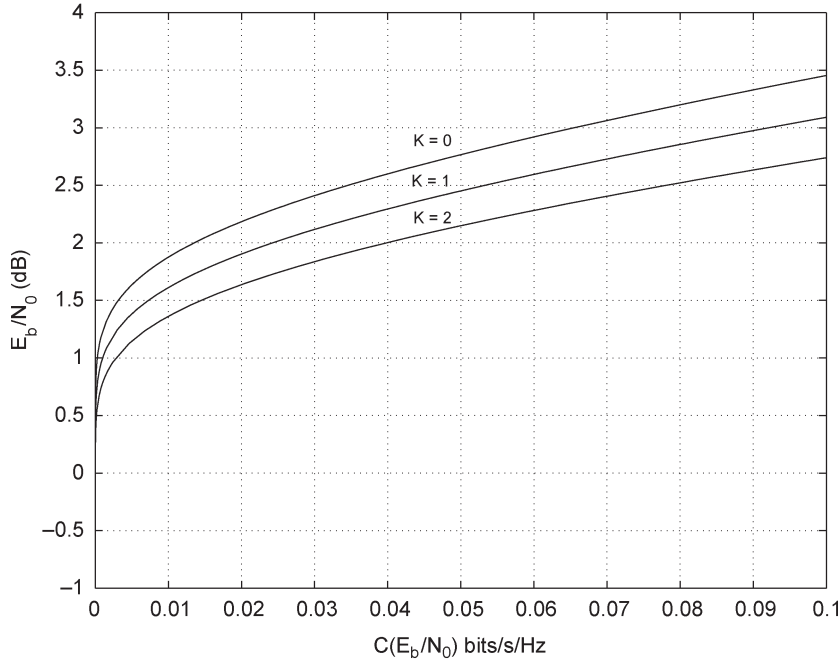


Fig. 4. E_b/N_0 (dB) versus spectral efficiency $C(E_b/N_0)$ (bits per second per hertz) for the Rician fading channel with phase noise with $K = 0, 1$, and 2.

and

$$\lim_{\text{SNR} \rightarrow 0} \frac{E \{ |x_{\text{SNR}}|^2 \mathbf{1} \{ |x_{\text{SNR}}|^2 > \nu - \epsilon \} \}}{E \{ |x_{\text{SNR}}|^2 \}} = 1 \quad \forall \epsilon > 0 \quad (31)$$

is necessary to achieve the minimum bit energy, and hence, the wideband slope.

Proof: See Appendix. ■

Remark 6: The minimum bit energy decreases to -1.59 dB as we approach the unfaded Gaussian channel, i.e., $K \rightarrow \infty$; or the peak constraint is relaxed, i.e., $\nu \rightarrow \infty$. We also note that for the Rayleigh case where $K = 0$, the minimum bit-energy expression can easily be obtained as a special case of the capacity per unit energy result of [15]. The wideband slope is zero for $\eta \geq 1$ as in the average-power-limited case, and hence, we conclude that achieving the minimum bit energy is extremely demanding in bandwidth.

Remark 7: It is also interesting to note that as $\eta \downarrow 0$ while keeping the fading variance γ^2 fixed (i.e., $\nu \downarrow 0$ or $N_0 \uparrow \infty$), $E_b^x/N_{0\text{min}} \rightarrow (1 + (1/K)) \log_e 2$ and $S_0 \rightarrow 2K^2/(1 + K)^2$. We notice that the limiting values are the expressions for the bit energy at zero spectral efficiency and wideband slope with $\kappa = 0$ in the limited peak-to-average power ratio case.

Remark 8: A class of input signals that satisfy the conditions (30) and (31) are identified to be first-order optimal, thereby achieving the minimum bit energy. Noting that these conditions are also necessary to achieve the wideband slope, we show in the Appendix that the two-mass-point amplitude distribution

$$F(|x|) = \left(1 - \frac{P_{\text{av}}}{\nu}\right) u(|x|) + \frac{P_{\text{av}}}{\nu} u(|x| - \sqrt{\nu}) \quad (32)$$

achieves both the minimum bit energy and the wideband slope. Indeed, a recent independent analysis by Huang and Meyn [10]

has shown that the two-mass-point distribution with one mass at the peak level and the other at the origin is capacity achieving for sufficiently low SNR values for a general class of channels, including the Rician channel, with fixed peak constraints.

C. Average-Power-Limited Rician Channel With Phase Noise

Finally, we comment on the spectral efficiency of the average-power-limited Rician fading channel with phase noise, which is introduced in [1]. Lapidoth and Shamai [5] have proven that for a general class of average-power-limited single-input single-output fading channels, the bit energy at zero spectral efficiency is $\log_e 2 = -1.59$ dB. If there is imperfect receiver-side information, Verdú [4] has shown that the wideband slope is zero. Fig. 4 plots the spectral-efficiency/bit-energy function for the noncoherent Rician channel with phase noise for $K = 0, 1$, and 2. Indeed, we observe that for all K , the bit-energy curves are approaching -1.59 dB with zero wideband slope.

IV. EFFICIENT SIGNALING IN THE LOW-POWER REGIME

Having analyzed the spectral-efficiency/bit-energy tradeoff in the low-power regime, we have seen that if the input is subject to second- and fourth-moment constraints (6) and (7), or average- and peak-power constraints (6) and (21) with $\kappa \leq (1 + K)^2$, then the wideband slope is positive and the numerical results indicate that for a large-enough Rician factor K , the minimum bit energy is achieved at zero spectral efficiency. Furthermore, if the noise spectral density N_0 and the peak-power constraint is fixed as the average power varies, the capacity curve is a concave function of the SNR, and hence, the minimum bit energy is also achieved at zero spectral efficiency.

Motivated by these observations, we will now investigate efficient signaling schemes in the low-power regime when the input has limited peakedness. Verdú [4] defines an input distribution to be first-order optimal if it satisfies the input constraints and achieves the first derivative of the capacity at zero SNR, and second-order optimal if, in addition, it achieves the second derivative of the capacity at zero SNR. So, a first-order optimal input achieves the energy per bit at zero spectral efficiency (which as noted before, need not be the minimum energy per bit), and a second-order optimal input achieves both the bit energy at zero spectral efficiency and the wideband slope. We have observed in [1] that for the noncoherent Rician fading channel with second- and fourth-moment input constraints, a particular two-mass-point input distribution (20) is capacity achieving for sufficiently small SNR. Based on this observation, we define the following signaling schemes, which overlay phase-shift keying on ON-OFF keying (OOK).

Definition 1: An ON-OFF binary phase-shift keying (OOBPSK) signal, parametrized by $0 < p \leq 1$, has the following constellation points with the corresponding probabilities

$$\begin{aligned} x_1 &= 0 && \text{with prob. } 1-p \\ x_2 &= +\sqrt{\frac{P_{\text{av}}}{p}} && \text{with prob. } \frac{p}{2} \\ x_3 &= -\sqrt{\frac{P_{\text{av}}}{p}} && \text{with prob. } \frac{p}{2} \end{aligned} \quad (33)$$

where P_{av} is the average power of the signal.

Definition 2: An ON-OFF quaternary phase-shift keying (OOQPSK) signal, parametrized by $0 < p \leq 1$, has the following constellation points with the corresponding probabilities

$$\begin{aligned} x_1 &= 0 && \text{with prob. } 1-p \\ x_i &= \sqrt{\frac{P_{\text{av}}}{2p}}(\pm 1 \pm j) && \text{with prob. } \frac{p}{4}, \quad i = 2, 3, 4, 5 \end{aligned} \quad (34)$$

where P_{av} is the average power of the signal.

From the above definitions, we immediately notice that $1/p$ is the kurtosis of the signals and having $p = 1$ reduces the signaling schemes to ordinary binary phase-shift keying (BPSK) and quaternary phase-shift keying (QPSK), respectively. Next, we investigate the performance of these schemes in the wideband regime when the input amplitude is subject to second- and fourth-moment limitations.

Proposition 4: For the Rician fading channel (5), with input constraints (6) and (7), an OOBPSK input with average power P_{av} and $1/\kappa \leq p \leq 1$ is first-order optimal. Furthermore, the first and second derivatives at zero SNR of the mutual information achieved by this input are given by

$$\dot{I}(0) = |m|^2 \text{ and } \ddot{I}(0) = \kappa\gamma^4 - (|m|^2 + \gamma^2)^2 - |m|^4 \quad (35)$$

respectively.

Proof: First note that an OOBPSK input (33) with average power P_{av} and $1/\kappa \leq p \leq 1$ satisfies the input constraints (6) and (7). The input-output mutual information is given by

$$I(x, y) = \int_{\mathcal{C}} \int_{\mathcal{C}} f_{y|x}(y|x) \ln \frac{f_{y|x}(y|x)}{f_y(y)} dy dF(x) \quad (36)$$

where $dF(x) = (1-p)\delta(x) + (p/2)\delta(x - \sqrt{P_{\text{av}}/p}) + (p/2)\delta(x + \sqrt{P_{\text{av}}/p})$, and $f_{y|x}$ is given in [1, eq. (5)]. Direct differentiation of (36) with respect to SNR gives (35). Achieving the first derivative of the capacity at zero SNR, OOBPSK input is first-order optimal. ■

Proposition 5: For the Rician fading channel model (5), with input constraints (6) and (7), the OOQPSK input with average power P_{av} and $p = 1/\kappa$ is second-order optimal, i.e., the first and second derivatives at zero SNR of the mutual information achieved by this input are given by

$$\dot{I}(0) = |m|^2 \text{ and } \ddot{I}(0) = \kappa\gamma^4 - (|m|^2 + \gamma^2)^2 \quad (37)$$

respectively.

Proof: The steps in the proof are essentially the same as in the proof of Proposition 4. The OOQPSK input (34) with average power P_{av} and $p = 1/\kappa$ satisfies the input constraints (6) and (7), and the input distribution in the mutual-information expression (36) now becomes

$$\begin{aligned} dF(x) &= (1-p)\delta(x) + \frac{p}{4}\delta\left(x - \sqrt{\frac{P_{\text{av}}}{2p}}(1+j)\right) \\ &+ \frac{p}{4}\delta\left(x - \sqrt{\frac{P_{\text{av}}}{2p}}(1-j)\right) \\ &\times \frac{p}{4}\delta\left(x - \sqrt{\frac{P_{\text{av}}}{2p}}(-1+j)\right) \\ &+ \frac{p}{4}\delta\left(x - \sqrt{\frac{P_{\text{av}}}{2p}}(-1-j)\right). \end{aligned} \quad (38)$$

Similarly, direct differentiation of the mutual information with respect to SNR provides (37). Achieving both the first and the second derivatives of the capacity at zero SNR, OOQPSK input is second-order optimal. ■

Remark 9: Note that the results of Propositions 4 and 5 hold when the fourth-moment constraint (7) is replaced by a peak-power constraint (21).

Remark 10: An interesting observation is that OOBPSK with $p \in [1/\kappa, 1]$ is, to first order, no better than ordinary BPSK (i.e., OOBPSK with $p = 1$) because the first derivative does not depend on the kurtosis of the signal. Since no information transfer takes place with phase modulation over the unknown Rayleigh fading channel, this explains why reliable communication over this channel is not possible in the asymptotic regime of zero spectral efficiency. For second order optimality, we need OOQPSK signaling with $p = 1/\kappa$. Therefore, in the low-power regime, we need both amplitude- and phase-modulation schemes in order to be spectrally efficient.

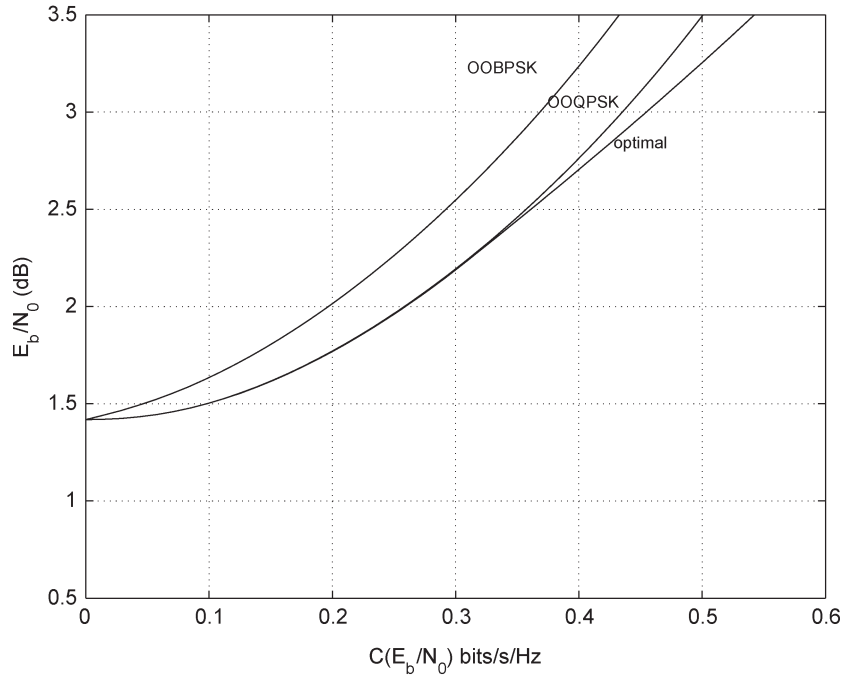


Fig. 5. E_b/N_0 (dB) versus spectral efficiency $C(E_b/N_0)$ (bits per second per hertz) curves for the optimal, OOQPSK, and OOBPSK signaling with $p = 1/\kappa$ in the Rician channel where $K = 1$ and $\kappa = 4$.

For the average-power-limited unfaded additive white Gaussian noise (AWGN) channel, Verdú [4] has shown that ordinary QPSK, which has unit kurtosis, is second-order optimal. Hence, for the unfaded channel, introducing an additional peak or fourth-moment constraint on the channel input does not degrade the performance in the wideband regime.

As a natural next step, we look at the wideband slopes achieved by OOBPSK and OOQPSK signaling.

Corollary 2: In the Rician fading channel (5), with input constraints (6) and (7), OOBPSK and OOQPSK signaling schemes with $p = 1/\kappa$ achieve the following wideband slopes:

$$\begin{aligned}
 S_{0,\text{OOBPSK}} &= \frac{2K^2}{K^2 + (1 + K)^2 - \kappa} \text{ and} \\
 S_{0,\text{OOQPSK}} &= \frac{2K^2}{(1 + K)^2 - \kappa}
 \end{aligned} \tag{39}$$

respectively.

Since the OOQPSK is second-order optimal, it achieves the optimal wideband slope. Note also that, for fixed κ and K , when both slopes are positive, OOBPSK achieves a smaller slope than that of OOQPSK. Hence, for the same bit energy and rate, OOBPSK needs more bandwidth in the low-power regime, or equivalently, for the same bandwidth and rate, it requires more bit energy. We need to be careful when the slopes are negative because in this case, we want to avoid operating in the very-low-power regime, as discussed previously.

Fig. 5 plots the spectral-efficiency–bit-energy curve for optimal, OOQPSK, and OOBPSK signaling, with $p = 1/\kappa$ in the Rician fading channel ($K = 1$), when the input is subject to $E\{|x|^2\} \leq P_{\text{av}}$ and $E\{|x|^4\} \leq \kappa P_{\text{av}}^2$, with $\kappa = 4$. Note that for this value of κ , the wideband slope is positive. Both

OOBPSK and OOQPSK achieve the minimum bit energy (first-order optimality). Being second-order optimal, OOQPSK also achieves the wideband slope and is very close to the optimal curve in the low-SNR regime. Therefore, we conclude that OOQPSK signaling is optimally efficient in the low-power limit. Note that OOBPSK achieves a smaller slope and hence, for fixed rate and power, it requires more bandwidth. In Fig. 6, κ is increased to 10. In this case, the wideband slope is negative and the minimum bit energy is achieved at a nonzero spectral efficiency, which implies that the very-low-power regime ought to be avoided. However, we observe that OOQPSK is still an efficient scheme, achieving very close to the minimum bit energy. Moreover, Fig. 7 shows how we increase the wideband slope when we use OOQPSK signaling with higher and higher kurtosis κ as long as $\kappa \leq (1 + K)^2$. Finally, for the Rayleigh fading channel, Fig. 8 plots the bit-energy/spectral-efficiency curve for the optimal and OOK signaling, which is in the form given by (20), for $\kappa = 2, 5$, and 10. We observe that OOK is an efficient signaling scheme, achieving the minimum bit energy in the cases of $\kappa = 5$ and 10. For $\kappa = 2$, the minimum bit energy is again achieved by OOK, however, as discussed in Section III-A, with $E\{|x|^2\} < P_{\text{av}}$. Therefore, in this case, OOK signaling in the form (20) is suboptimal, achieving a higher minimum bit energy.

Up to now, we have considered the cases in which the fourth-moment or the peak-power constraint varies with the average power of the signal. When there is a fixed peak limit, we have seen in Section III-B.2 that the two-mass-point input amplitude distribution (32) achieves the minimum bit energy. Therefore, when the input is subject to (6) and (28), OOK signaling, where the ON level is at $\sqrt{\nu}$ with probability P_{av}/ν , is first-order optimal. Moreover, it can be easily seen that this signaling scheme is second-order optimal in the Rayleigh channel and

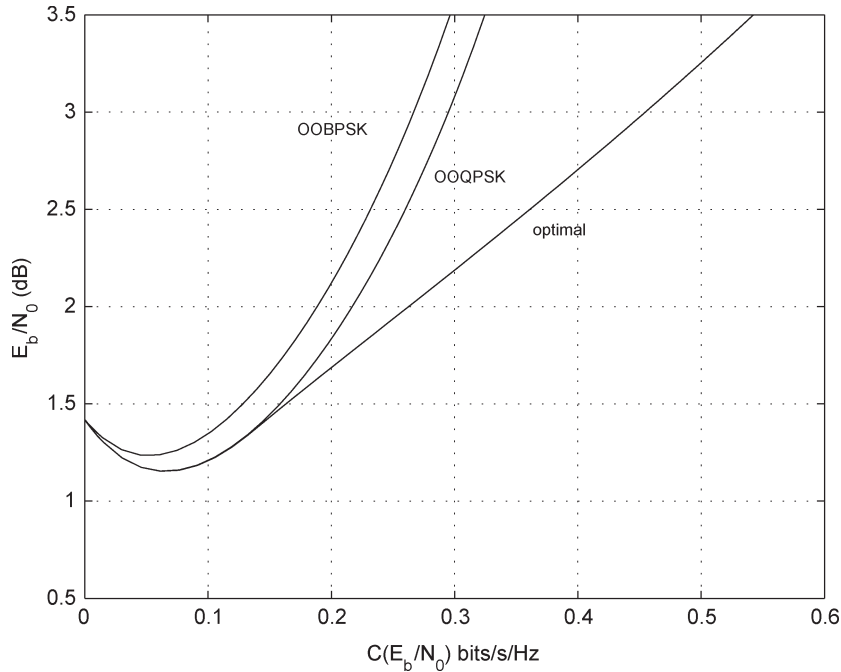


Fig. 6. E_b/N_0 (dB) versus spectral efficiency $C(E_b/N_0)$ (bits per second per hertz) curves for optimal, OOQPSK, and OOBPSK signaling with $p = 1/\kappa$ in the Rician Channel where $K = 1$ and $\kappa = 10$.

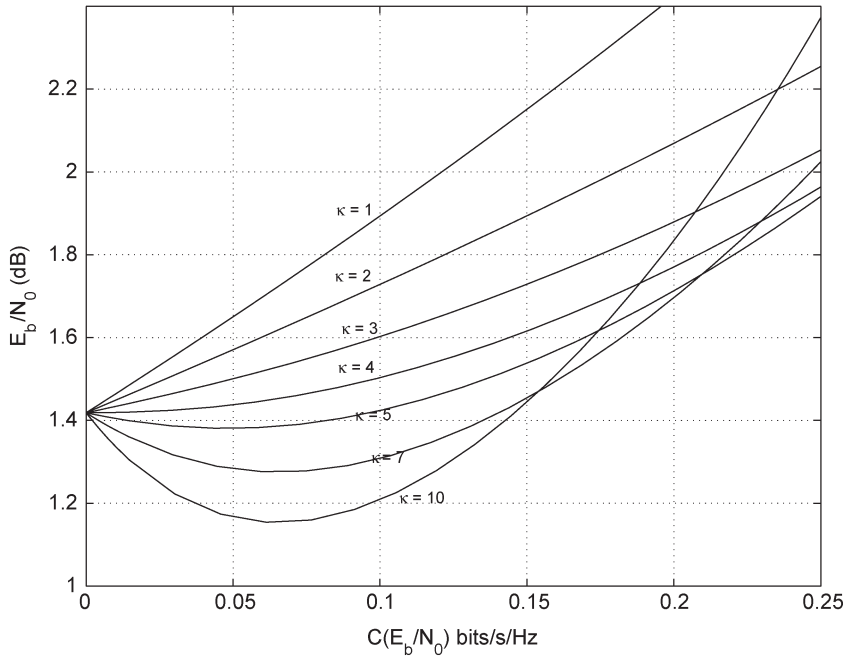


Fig. 7. E_b/N_0 (dB) versus spectral efficiency $C(E_b/N_0)$ (bits per second per hertz) curves for OOQPSK signaling for $1/p = \kappa = 1, 2, 3, 4, 5, 7,$ and 10 in the Rician channel with $K = 1$.

in the Rician channel when the wideband slope is zero, i.e., $\nu \geq N_0/\gamma^2$. We also investigate the low-power performance of OOQPSK signaling. Note that by choosing $p = P_{av}/\nu$, we obtain OOQPSK signals whose ON-level amplitudes are fixed at $\sqrt{\nu}$. Since the proof of the following result is similar to those of Propositions 4 and 5, and only involves differentiating the mutual information with respect to the SNR, it is omitted.

Proposition 6: For the Rician fading channel (5), with input constraints (6) and (28), and fixed noise density N_0 , the OOQPSK input with average power P_{av} and $p = P_{av}/\nu$ is first-

order optimal. Furthermore, the first and second derivatives at zero SNR of the mutual information achieved by this input are given by

$$\begin{aligned} \dot{I}(0) &= (|m|^2 + \gamma^2) - N_0 \frac{\log_e \left(\frac{\gamma^2}{N_0} \nu + 1 \right)}{\nu} \\ \ddot{I}(0) &= \begin{cases} \frac{N_0^2}{\nu^2} \left(1 - \frac{1}{16} \sum_{i,j=2}^5 \int \frac{f_{y|x=x_i} f_{y|x=x_j}}{f_{y|x=0}} dy \right), & \nu < \frac{N_0}{\gamma^2} \\ -\infty, & \nu \geq \frac{N_0}{\gamma^2} \end{cases} \end{aligned}$$

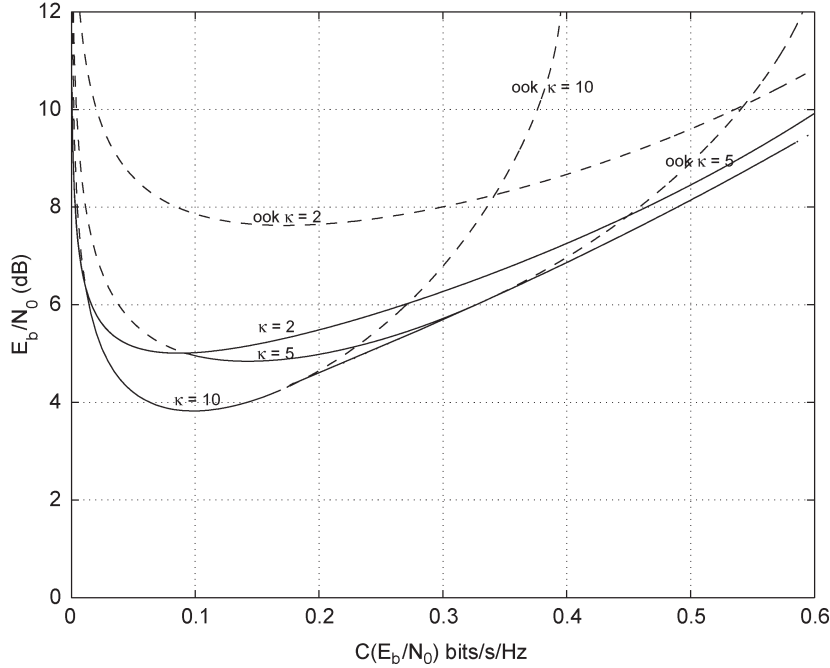


Fig. 8. E_b/N_0 (dB) versus spectral efficiency $C(E_b/N_0)$ (bits per second per hertz) for the optimal (solid curves) and the OOK signaling (dashed curves) with $\kappa = 2, 5,$ and 10 in the Rayleigh channel.

respectively, where $f_{y|x=a} = [1/\pi(\gamma^2|a|^2 + N_0)]\exp[(-|y - ma|^2)/(\gamma^2|a|^2 + N_0)]$.

Remark 11: We note that the OOQPSK signaling is second-order optimal in the Rician channel only when the wideband slope is zero, i.e., $\nu \geq N_0/\gamma^2$, and achieves a smaller slope if $\nu < N_0/\gamma^2$.

V. CONCLUSION

For the noncoherent Rician fading channel, we have analyzed the spectral-efficiency/bit-energy tradeoff in the low-power regime when the input peakedness is limited by a fourth-moment or a peak-power constraint. We have found analytical expressions for the bit energy required at zero spectral efficiency and wideband slope.

We first considered the case in which the input, in addition to the average-power constraint, is subject to a fourth-moment constraint $E\{|x|^4\} \leq \kappa P_{av}^2$. We have shown that if $\kappa > (1 + K)^2$, the wideband slope is negative. Hence, the minimum bit energy is achieved at some nonzero spectral efficiency C^* . In this case, we have identified a forbidden region where one should not operate. In this region where $C < C^*$, decreasing the spectral efficiency further (i.e., increasing the bandwidth for fixed-rate transmission) increases the bit energy required for reliable communications. Indeed, for the unknown Rayleigh fading channel, the bit energy at zero spectral efficiency is infinite.

If $\kappa \leq (1 + K)^2$, the wideband slope is positive and we have conjectured that for a large-enough Rician factor K , the minimum bit energy is achieved at zero spectral efficiency. This bit energy can be achieved by BPSK signaling.

We have also analyzed the case where the input peakedness is restricted by a peak-power constraint. If the input peak-to-average power ratio is limited, i.e., the input is subject to $|x|^2 \leq \kappa P_{av}$, the same expressions for the bit energy at zero

spectral efficiency and wideband slope are found as in the fourth-moment-limited case. Therefore, the same conclusions as above are drawn. If the input is subject to a fixed peak limit, i.e., $|x|^2 \leq \nu$, we have obtained the minimum bit energy and wideband slope. We have shown that if $(\gamma^2/N_0)\nu \geq 1$, then the wideband slope is zero, and hence, achieving the minimum bit energy is very demanding in bandwidth.

We have defined the OOBPSK and OOQPSK signaling schemes and analyzed their low-power performance. We have shown that when the input is subject to $E\{|x|^4\} \leq \kappa P_{av}^2$ or $|x|^2 \leq \kappa P_{av}$, OOQPSK is second-order optimal while OOBPSK is first-order optimal. Therefore, achieving the optimal wideband slope, OOQPSK signaling turns out to be a very efficient scheme in the low-power regime. In the case of the fixed peak limit, we have seen that OOK signaling with ON level fixed at the peak level achieves the minimum bit energy.

APPENDIX

A. Proof of Proposition 3

Since the capacity curve is a concave function of the SNR in this case, the minimum received bit energy is achieved at zero spectral efficiency and can be obtained from $E_b^r/N_{0,\min} = (|m|^2 + \gamma^2) \log_e 2 / \dot{C}(0)$. $\dot{C}(0)$ is easily found using the following formula [19]

$$\begin{aligned} \dot{C}(0) &= N_0 \sup_{|x_0| \leq \nu} \frac{D(f_{y|x=x_0} \| f_{y|x=0})}{|x_0|^2} \\ &= N_0 \sup_{|x_0| \leq \nu} \frac{\frac{|m|^2 + \gamma^2}{N_0} |x_0|^2 - \log_e \left(\frac{\gamma^2}{N_0} |x_0|^2 + 1 \right)}{|x_0|^2} \end{aligned} \quad (40)$$

$$= |m|^2 + \gamma^2 - N_0 \frac{\log_e \left(\frac{\gamma^2}{N_0} \nu + 1 \right)}{\nu}. \quad (41)$$

Next, we show that input signaling that satisfies (30) and (31) is required to achieve the minimum bit energy, and hence, the optimal wideband slope. We adopt an approach similar to that of [4], where flash signaling is shown to be necessary to achieve the minimum bit energy in the absence of the peak constraint (28).

- 1) We first prove that signaling that satisfies (30) and (31) achieves the first derivative of the capacity. Note that, in general, we have

$$\dot{I}(0) = \lim_{\text{SNR} \rightarrow 0} \frac{I(x_{\text{SNR}}; y)}{\text{SNR}} \quad (42)$$

$$= \lim_{\text{SNR} \rightarrow 0} N_0 \frac{E \{D(f_{y|x=x_{\text{SNR}}} \| f_{y|x=0})\}}{E \{|x_{\text{SNR}}|^2\}} \quad (43)$$

$$= |m|^2 + \gamma^2 - \lim_{\text{SNR} \rightarrow 0} N_0 \frac{E \left\{ \log_e \left(\frac{\gamma^2}{N_0} |x_{\text{SNR}}|^2 + 1 \right) \right\}}{E \{|x_{\text{SNR}}|^2\}}. \quad (44)$$

Fix some $\epsilon \in (0, \nu)$. Then, we can write

$$\begin{aligned} & E \left\{ \log_e \left(\frac{\gamma^2}{N_0} |x_{\text{SNR}}|^2 + 1 \right) \right\} \\ &= E \left\{ \log_e \left(\frac{\gamma^2}{N_0} |x_{\text{SNR}}|^2 + 1 \right) 1 \{|x_{\text{SNR}}|^2 > \nu - \epsilon\} \right\} \\ &+ E \left\{ \log_e \left(\frac{\gamma^2}{N_0} |x_{\text{SNR}}|^2 + 1 \right) 1 \{|x_{\text{SNR}}|^2 \leq \nu - \epsilon\} \right\}. \end{aligned} \quad (45)$$

Using $\log_e(1+x) \leq x$, we have

$$\begin{aligned} & \lim_{\text{SNR} \rightarrow 0} \frac{E \left\{ \log_e \left(\frac{\gamma^2}{N_0} |x_{\text{SNR}}|^2 + 1 \right) 1 \{|x_{\text{SNR}}|^2 \leq \nu - \epsilon\} \right\}}{E \{|x_{\text{SNR}}|^2\}} \\ & \leq \frac{\gamma^2}{N_0} \lim_{\text{SNR} \rightarrow 0} \frac{E \{|x_{\text{SNR}}|^2 1 \{|x_{\text{SNR}}|^2 \leq \nu - \epsilon\}\}}{E \{|x_{\text{SNR}}|^2\}} \\ & = 0 \end{aligned} \quad (46)$$

where (46) follows from (31). Moreover, using the fact that $\log_e(1+x)/x$ is monotonically decreasing, $|x_{\text{SNR}}|^2 \stackrel{\text{a.s.}}{\leq} \nu$, and (31), we easily observe that

$$\begin{aligned} & \frac{\log_e \left(\frac{\gamma^2}{N_0} \nu + 1 \right)}{\nu} \\ & \leq \lim_{\text{SNR} \rightarrow 0} \frac{E \left\{ \log_e \left(\frac{\gamma^2}{N_0} |x_{\text{SNR}}|^2 + 1 \right) 1 \{|x_{\text{SNR}}|^2 > \nu - \epsilon\} \right\}}{E \{|x_{\text{SNR}}|^2\}} \\ & \leq \frac{\log_e \left(\frac{\gamma^2}{N_0} (\nu - \epsilon) + 1 \right)}{\nu - \epsilon}. \end{aligned} \quad (47)$$

From (44)–(47), we have

$$\begin{aligned} & |m|^2 + \gamma^2 - N_0 \frac{\log_e \left(\frac{\gamma^2}{N_0} (\nu - \epsilon) + 1 \right)}{\nu - \epsilon} \\ & \leq \dot{I}(0) \\ & \leq |m|^2 + \gamma^2 - N_0 \frac{\log_e \left(\frac{\gamma^2}{N_0} \nu + 1 \right)}{\nu}. \end{aligned} \quad (48)$$

Since ϵ is arbitrary, we conclude that input signaling satisfying (30) and (31) achieves $\dot{C}(0)$.

- 2) Now, we will show that (30) and (31) are necessary conditions to achieve $\dot{C}(0)$. Note that (30) is dictated by the input constraints (6) and (28). Using again the monotonicity of $\log_e(1+x)/x$, we observe, for arbitrary $\epsilon \in (0, \nu)$, that

$$\begin{aligned} & E \left\{ \log_e \left(\frac{\gamma^2}{N_0} |x_{\text{SNR}}|^2 + 1 \right) 1 \{|x_{\text{SNR}}|^2 > \nu - \epsilon\} \right\} \\ & \geq \frac{\log_e \left(\frac{\gamma^2}{N_0} \nu + 1 \right)}{\nu} \\ & \quad \times E \{|x_{\text{SNR}}|^2 1 \{|x_{\text{SNR}}|^2 > \nu - \epsilon\}\} \\ & E \left\{ \log_e \left(\frac{\gamma^2}{N_0} |x_{\text{SNR}}|^2 + 1 \right) 1 \{|x_{\text{SNR}}|^2 \leq \nu - \epsilon\} \right\} \\ & \geq \frac{\log_e \left(\frac{\gamma^2}{N_0} (\nu - \epsilon) + 1 \right)}{\nu - \epsilon} \\ & \quad \times E \{|x_{\text{SNR}}|^2 1 \{|x_{\text{SNR}}|^2 \leq \nu - \epsilon\}\}. \end{aligned} \quad (49)$$

Combining (44), (45), (49), and (50), we have

$$\begin{aligned} \dot{I}(0) & \leq |m|^2 + \gamma^2 - \frac{\log_e \left(\frac{\gamma^2}{N_0} \nu + 1 \right)}{\nu} \\ & \quad - \left[\frac{\log_e \left(\frac{\gamma^2}{N_0} (\nu - \epsilon) + 1 \right)}{\nu - \epsilon} - \frac{\log_e \left(\frac{\gamma^2}{N_0} \nu + 1 \right)}{\nu} \right] \\ & \quad \times \lim_{\text{SNR} \rightarrow 0} \frac{E \{|x_{\text{SNR}}|^2 1 \{|x_{\text{SNR}}|^2 \leq \nu - \epsilon\}\}}{E \{|x_{\text{SNR}}|^2\}} \end{aligned} \quad (51)$$

from which we notice that the condition $\lim_{\text{SNR} \rightarrow 0} [E \{|x_{\text{SNR}}|^2 1 \{|x_{\text{SNR}}|^2 \leq \nu - \epsilon\}\} / E \{|x_{\text{SNR}}|^2\}] = 0$ for all $0 < \epsilon < \nu$, which is equivalent to (31), is required to achieve $\dot{C}(0)$.

- 3) In this part, we obtain the optimal wideband slope by evaluating $\ddot{C}(0)$. For the input x_{SNR} that achieves both $\dot{C}(0)$ and $\ddot{C}(0)$, we can write

$$\begin{aligned} \ddot{C}(0) &= 2 \lim_{\text{SNR} \rightarrow 0} \frac{I(x_{\text{SNR}}; y) - \dot{C}(0)\text{SNR}}{\text{SNR}^2} \\ &= -2 \lim_{\text{SNR} \rightarrow 0} \frac{D(f_{y|x=x_{\text{SNR}}})}{\text{SNR}^2}. \end{aligned} \quad (52)$$

Furthermore, by [1, Proposition 1], we can assume, without loss of optimality, that x_{SNR} has a uniformly distributed phase independent of the amplitude. With this assumption, it can be easily verified that

$$D(f_y \| f_{y|x=0}) = D(f_R \| f_{R|r=0}) \quad (53)$$

where, as in [1], $R = |y|^2/N_0$ and $r = (\gamma/\sqrt{N_0})x_{\text{SNR}}$, and therefore $f_R = \int_0^\infty g(R, r) dF_r(r)$ with $g(R, r) = (1/(1+r^2)) \exp[-(R+Kr^2)/(1+r^2)] I_0[(2\sqrt{K}r\sqrt{R})/(1+r^2)]$, and $f_{R|r=0} = \exp(-R)$. Following the approach employed in the proof of [4, Th. 16], we write

$$\begin{aligned} D(f_R \| f_{R|r=0}) &= E \{ (1 + \text{SNR}(W + V)) \log_e (1 + \text{SNR}(W + V)) \} \\ & \quad (54) \end{aligned}$$

where

$$\begin{aligned} V &= \frac{P(r^2 > \eta - \epsilon)}{\text{SNR}} \left(\frac{\tilde{f}_R}{f_{R|r=0}} - 1 \right) \\ W &= \frac{1 - P(r^2 > \eta - \epsilon)}{\text{SNR}} \left(\frac{\hat{f}_R}{f_{R|r=0}} - 1 \right). \quad (55) \end{aligned}$$

In the above formulation, \tilde{f}_R and \hat{f}_R are the distributions of R conditioned on $r^2 > \eta - \epsilon$ and $r^2 \leq \eta - \epsilon$, respectively, for some fixed $\epsilon \in (0, \eta)$. Using the facts that $(x + 1) \log_e(1 + x) = x + (1/2)x^2 + o(x^2)$, and $(V + W)$ has zero mean and converges to a nonzero random variable for vanishing SNR when (30) and (31) are satisfied, we have

$$\begin{aligned} \lim_{\text{SNR} \rightarrow 0} \frac{D(f_R \| f_{R|r=0})}{\text{SNR}^2} &= \lim_{\text{SNR} \rightarrow 0} \frac{E \{ (1 + \text{SNR}(W + V)) \log_e (1 + \text{SNR}(W + V)) \}}{\text{SNR}^2} \\ & \quad (56) \end{aligned}$$

$$= \frac{1}{2} \lim_{\text{SNR} \rightarrow 0} E \{ (W + V)^2 \}. \quad (57)$$

Noting that (30) and (31) are necessary to achieve the minimum bit energy, and hence, the optimal wideband

slope, we will first consider $E\{V^2\}$

$$\begin{aligned} \lim_{\text{SNR} \rightarrow 0} E\{V^2\} &= \lim_{\text{SNR} \rightarrow 0} \frac{P^2(r^2 > \eta - \epsilon)}{\text{SNR}^2} E \left\{ \frac{\tilde{f}_R^2}{f_{R|r=0}^2} - 1 \right\} \\ & \quad (58) \end{aligned}$$

$$\geq \frac{N_0^2}{\nu^2} \lim_{\text{SNR} \rightarrow 0} E \left\{ \frac{\tilde{f}_R^2}{f_{R|r=0}^2} - 1 \right\} \quad (59)$$

$$= \frac{N_0^2}{\nu^2} \lim_{\text{SNR} \rightarrow 0} \left(\int e^R \tilde{f}_R^2 dR - 1 \right) \quad (60)$$

$$\begin{aligned} &\geq \frac{N_0^2}{\nu^2} \left(\int_{\Omega_1} e^R g^2(R, \sqrt{\eta}) dR \right. \\ & \quad \left. + \int_{\mathbb{R}^+ \setminus \Omega_1} e^R g^2(R, \sqrt{\eta - \epsilon}) dR - 1 \right) \quad (61) \end{aligned}$$

where $\Omega_1 = \{R : g(R, \sqrt{\eta}) \leq g(R, \sqrt{\eta - \epsilon})\}$, and $\mathbb{R}^+ = [0, \infty)$. Equation (59) follows by assuming (31) and noting

$$\begin{aligned} 1 &= \lim_{\text{SNR} \rightarrow 0} \frac{E\{|x_{\text{SNR}}|^2\} 1\{|x_{\text{SNR}}|^2 > \nu - \epsilon\}}{E\{|x_{\text{SNR}}|^2\}} \\ &= \lim_{\text{SNR} \rightarrow 0} \frac{P(|x_{\text{SNR}}|^2 > \nu - \epsilon) E\{|x_{\text{SNR}}|^2\} | |x_{\text{SNR}}|^2 > \nu - \epsilon\}}{E\{|x_{\text{SNR}}|^2\}} \\ &\leq \nu \lim_{\text{SNR} \rightarrow 0} \frac{P(|x_{\text{SNR}}|^2 > \nu - \epsilon)}{E\{|x_{\text{SNR}}|^2\}} \quad (62) \end{aligned}$$

$$= \frac{\nu}{N_0} \lim_{\text{SNR} \rightarrow 0} \frac{P(|x_{\text{SNR}}|^2 > \nu - \epsilon)}{\text{SNR}}. \quad (63)$$

Equation (61) follows by noting that

$$\tilde{f}_r = \int g(R, r) d\tilde{F}_r(r) \quad (64)$$

$$\geq \min \{g(R, \sqrt{\eta}), g(R, \sqrt{\eta - \epsilon})\} \quad (65)$$

where \tilde{F}_r is the distribution of r conditioned on $r^2 > \eta - \epsilon$. Equation (65) follows from the fact that $g(R, r)$ is either a monotonically decreasing or a first monotonically increasing and then decreasing function of r .

$$\ddot{C}(0) = \begin{cases} -\frac{N_0^2}{\nu^2} \left(\frac{1}{1 - \frac{\gamma^4 \nu^2}{N_0^2}} \exp \left(\frac{2|m|^2 \gamma^2 \nu^2}{1 - \frac{\gamma^4 \nu^2}{N_0^2}} \right) I_0 \left(\frac{2|m|^2 \nu}{1 - \frac{\gamma^4 \nu^2}{N_0^2}} \right) - 1 \right), & \nu < \frac{N_0}{\gamma^2} \\ -\infty, & \nu \geq \frac{N_0}{\gamma^2} \end{cases} \quad (69)$$

Similar analysis leads to $\lim_{\text{SNR} \rightarrow 0} E\{WV\} = 0$. Therefore from (52), (57), and (61), we have

$$\ddot{C}(0) \leq -\frac{N_0^2}{\nu^2} \left(\int_{\Omega_1} e^R g^2(R, \sqrt{\eta}) dR + \int_{\mathbb{R} \setminus \Omega_1} e^R g^2(R, \sqrt{\eta - \epsilon}) dR - 1 \right). \quad (66)$$

As $\epsilon \in (0, \eta)$ is arbitrary, we have

$$\ddot{C}(0) \leq -\frac{N_0^2}{\nu^2} \left(\int_0^\infty e^R g^2(R, \sqrt{\eta}) dR - 1 \right). \quad (67)$$

It can be easily shown that the two-mass-point input amplitude distribution

$$F(|x|) = \left(1 - \frac{P_{\text{av}}}{\nu}\right) u(|x|) + \frac{P_{\text{av}}}{\nu} u(|x| - \sqrt{\nu}) \quad (68)$$

achieves both $\dot{C}(0)$ and the upper bound in (67). Therefore, we conclude that (67) is indeed satisfied with equality, and therefore, we have (69), seen at the bottom of the previous page, where (69) is obtained by evaluating a closed-form expression for the integral in (67). The wideband slope is obtained by inserting (41) and (69) into (4).

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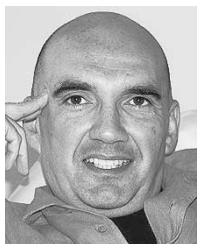
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