MECH 223 – Engineering Statics

Final Exam, December 17th 2015

Question 1 (20 + 5 points)

(a) (10 points) In each of the two following examples, in the column to the right, draw a free body diagram (FBD) of the body to be isolated, shown in the middle column. Dimensions and numerical values are omitted for simplicity.

<table>
<thead>
<tr>
<th>Description</th>
<th>Body to be Isolated</th>
<th>FBD</th>
</tr>
</thead>
<tbody>
<tr>
<td>Boom $OA$, of negligible mass compared with mass $m$. Boom hinged at $O$ and supported by hoisting cable at $B$.</td>
<td><img src="image1.png" alt="Boom Diagram" /></td>
<td><img src="image2.png" alt="FBD Diagram" /></td>
</tr>
<tr>
<td>Bent rod welded to support at $A$ and subjected to two forces and couple</td>
<td><img src="image3.png" alt="Bent Rod Diagram" /></td>
<td><img src="image4.png" alt="FBD Diagram" /></td>
</tr>
<tr>
<td>Lawn roller of mass $m$ being pushed up incline $\theta$ (assume no slipping of the roller)</td>
<td><img src="image5.png" alt="Lawn Roller Diagram" /></td>
<td><img src="image6.png" alt="FBD Diagram" /></td>
</tr>
<tr>
<td>Pry bar lifting body $A$ having smooth horizontal surface. Bar rests on horizontal rough surface.</td>
<td><img src="image7.png" alt="Pry Bar Diagram" /></td>
<td><img src="image8.png" alt="FBD Diagram" /></td>
</tr>
</tbody>
</table>
(b) (6 points) Draw the corresponding Free Body Diagrams for the three following cases

(c) (4 points) Explain why the triple product of three vectors is equal to zero if the vectors are coplanar.

**Answer:** Two possible explanations:

1. The first explanation is based on the physical interpretation of the triple product as the volume of the parallelepiped created by the three vectors. If all the vectors are coplanar, that volume is zero.

2. The second explanation utilizes properties of the cross and the dot product. We know that the cross product produces a vector perpendicular to the plane in which the two vectors lie. If all the vectors are coplanar, that means that the cross product is also perpendicular to the third vector. We know that the dot product of two perpendicular vectors is zero by definition ($\vec{B} \cdot \vec{C} = |\vec{B}| \cdot |\vec{C}| \cdot \cos \theta$ and $\cos 90^\circ = 0$). Which means that the triple product of coplanar vectors is zero.
Form the scalar product $\vec{B} \cdot \vec{C}$ and use the result to prove the identity

$$\cos(\alpha - \beta) = \cos \alpha \cos \beta + \sin \alpha \sin \beta$$

**Answer:** By definition the dot product of $\vec{B} \cdot \vec{C}$ is $|\vec{B}| \cdot |\vec{C}| \cdot \cos \theta$ where $\theta$ is the angle between the two vectors when they are joined tail to tail. In the case of the figure in this problem $\theta = \alpha - \beta \Rightarrow \vec{B} \cdot \vec{C} = |\vec{B}| \cdot |\vec{C}| \cdot \cos(\alpha - \beta)$. On the other hand we can do a dot product by Cartesian components of each vector. $\vec{B} = |\vec{B}|(\cos \alpha \cdot \vec{i} + \sin \alpha \cdot \vec{j})$ and $\vec{C} = |\vec{C}|(\cos \beta \cdot \vec{i} + \sin \beta \cdot \vec{j})$. $\Rightarrow \vec{B} \cdot \vec{C} = |\vec{B}| \cdot |\vec{C}|(\cos \alpha \cos \beta + \sin \alpha \sin \beta)$. Equating the two results produces $|\vec{B}| \cdot |\vec{C}| \cdot \cos(\alpha - \beta) = |\vec{B}| \cdot |\vec{C}|(\cos \alpha \cos \beta + \sin \alpha \sin \beta)$

$$\Rightarrow \cos(\alpha - \beta) = (\cos \alpha \cos \beta + \sin \alpha \sin \beta) \quad \text{QED}$$

**Question 2 (20 + 5 points)** For the shape in the drawing:

(a) (10 points) **By integration** find the $x$ location of the centroid (assume constant density). **Solutions by other means will not carry any points!**

**Solution:**

By plugging point $(a, 2b)$ into the parabolic formula $y = kx^2$; The formula for the straight line (top border of the triangle) is $y = -\frac{b}{a}x + 2b$.

We can use either a vertical or a horizontal strip. For a vertical strip, integration variable is $x$.

$$x_{el} = x$$

For the gray area: $dA = \frac{2b}{a^2} x^2 \, dx$; $A_1 = \int_0^a dA = \int_0^a \frac{2b}{a^2} x^2 \, dx = \left[ \frac{2b x^3}{3a^2} \right]_0^a = \frac{2}{3} ba$
For the triangle: $A = \left( -\frac{b}{a} x + 2b \right) dx$; $A_2 = \int_a^2 dx A = \int_a^2 \left( -\frac{b}{a} x + 2b \right) dx = \left[ -\frac{bx^2}{2a} + 2bx \right]_a^2 = -2ba + 4ba + \frac{1}{2}ba - 2ba = \frac{1}{2}ba$

$A = A_1 + A_2 = \frac{7}{6}ba$

$b) (10$ points) By treating the body as a composite body, find the $y$ location of the centroid (assume constant density). Solutions by other means will not carry any points! 

Solution:

$$\bar{x} = \frac{\int_0^2 \frac{2}{a} x^2 dx + \int_a^2 x \left( -\frac{b}{a} x + 2b \right) dx}{A} = \frac{\left[ \frac{2}{4a^2}x^4 \right]_0^a + \left[ -\frac{bx^3}{3} + \frac{2bx^2}{2} \right]_a^2}{\frac{7}{6}ba} = \frac{6\left\{ \frac{1}{2}ba^2 - \frac{8}{3}ba^2 + 4ba^2 + \frac{1}{3}ba^2 - ba^2 \right\}}{7ba} = a$$

(c) (Bonus 5 points) If the density of the triangle is twice as much as the density of the parabolic spandrel, find the distance between the centroid of the body and its center of mass.

Solution:

$$\rho_{\text{tri}} = 2\rho_{\text{par}} = 2\rho$$

$$\bar{y} = \frac{\sum_i \bar{y}_i A_i}{\sum_i A_i} = \frac{3 \cdot \frac{2}{10} \cdot a \cdot 2b + \frac{1}{3} \cdot \frac{2}{2}}{a \cdot 2b + \frac{ba}{2}} = \frac{4ab^2}{10} + \frac{ab^2}{6} = \frac{6 \cdot 17ab^2}{7 \cdot 30ab} = \frac{17}{35}b$$

$$\bar{y}_m = \frac{\sum_i \bar{y}_i m_i}{\sum_i m_i} = \frac{\bar{y}_{\text{par}} A_{\text{par}} \rho_{\text{par}} + \bar{y}_{\text{tri}} A_{\text{tri}} \rho_{\text{tri}}}{A_{\text{par}} \rho_{\text{par}} + A_{\text{tri}} \rho_{\text{tri}}} = \rho \left( \frac{3 \cdot \frac{2}{10} a \cdot 2b + \frac{1}{3} \cdot \frac{2}{2}}{5} + \frac{ab^2}{3} \right) = \frac{2ab^2 + \frac{ab^2}{5}}{5ab^2} = \frac{11}{25}b$$
Question 3 (30 points) A 120-lb cabinet is mounted on casters which can be locked to prevent their rotation. The coefficient of static friction between the floor and each caster is 0.30.

(a) (10 points) If \( h = 32 \) in., and if all casters are locked, determine the magnitude of the force \( P \) required to move the cabinet to the right. Show that the cabinet slides right rather than tips.

Solution:
(b) (10 points) If $h$ is no longer 32 in., for the force you calculated in (a), find the maximum $h$ before the cabinet tips.

Solution:

(c) (5 points) If $h=32$ in., find the force $P$ needed to slide the cabinet right if the casters at $B$ are locked and the casters at $A$ are free to rotate.

Solution:

(d) (5 points) If $h=32$ in., find the force $P$ needed to slide the cabinet right if the casters at $A$ are locked and the casters at $B$ are free to rotate.

Solution:
Question 4 (30 + 5 points) For the arched roof truss:

(a) (15 points) Determine the forces in members DE, EI, FI, and HI.

**Solution:**

\[
\sum M_A = 0 = (-25 \times 4 - 75 \times 10 - 100 \times 20 - 100 \times 30 - 25 \times 36 + G \times 40) \text{ kN} \times \text{m}
\]

\[
\Rightarrow \quad G = 168.75 \text{ kN}
\]

(b) (15 points) If the external force at E is replaced with a 100 kN force, find the forces in members DI, and DJ.

**Solution:**

From external analysis. Reactions at A and G are no longer equal because the loading is no longer symmetric.
\[ x = \tan^{-1} \frac{3}{10} = 16.699^\circ \; ; \; \beta = \tan^{-1} \frac{7}{4} = 60.255^\circ \]

\[ \sum M_D = 0 = (-100 \times 10 - 25 \times 16 + 168.75 \times 20 - IJ \times 7)kN \cdot m \]
\[ \Rightarrow \quad IJ = 282.14 \, kN \cdot T \]

\[ \sum M_E = 0 = (-25 \times 6 + 168.75 \times 10 - IJ \times 4 - ID \times (\sin \beta \times 6 + \cos \beta \times 4))kN \cdot m \quad \Rightarrow \quad ID = 56.844 \, kN \cdot T \]

\[ \sum M_C = 0 = (-100 \times 10 - 100 \times 20 - 25 \times 26 + 168.75 \times 30 - IJ \times 4 - DJ \times (\sin \beta \times 10 - \cos \beta \times 3))kN \cdot m \]
\[ \Rightarrow \quad DJ = 46.718 \, kN \cdot T \]

\[ DI = 56.84 \, kN \cdot T \; ; \; DJ = 46.72 \, kN \cdot T \]

(c) (Bonus 5 points) What is the basic difference between a truss and a frame? Why can’t the Method of Joints and Method of Sections be applied to frames?

**Solution:** Trusses are assumed to have only two force members, while frames have multi-force members. MoJ and MoS can’t be applied to frames because the direction of the forces in the multi-force members can no longer be assumed to be along the member.

**Show your work!**

**Good Luck!**
### Centroids of Common 2D Bodies

<table>
<thead>
<tr>
<th>Shape</th>
<th>$\bar{x}$</th>
<th>$\bar{y}$</th>
<th>Area</th>
</tr>
</thead>
<tbody>
<tr>
<td>Triangular area</td>
<td>$h/3$</td>
<td></td>
<td>$bh/2$</td>
</tr>
<tr>
<td>Quarter-circular area</td>
<td>$4r/3\pi$</td>
<td>$4r/3\pi$</td>
<td>$\pi r^2/4$</td>
</tr>
<tr>
<td>Semicircular area</td>
<td>0</td>
<td>$4r/3\pi$</td>
<td>$\pi r^2/2$</td>
</tr>
<tr>
<td>Quarter-elliptical area</td>
<td>$4a/3\pi$</td>
<td>$4b/3\pi$</td>
<td>$\pi ab/4$</td>
</tr>
<tr>
<td>Semicircular area</td>
<td>0</td>
<td>$4b/3\pi$</td>
<td>$\pi ab/2$</td>
</tr>
<tr>
<td>Semiparabolic area</td>
<td>$3a/8$</td>
<td>$3h/5$</td>
<td>$2ah/3$</td>
</tr>
<tr>
<td>Parabolic area</td>
<td>0</td>
<td>$3h/5$</td>
<td>$4ah/3$</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Shape</th>
<th>$\bar{x}$</th>
<th>$\bar{y}$</th>
<th>Area</th>
</tr>
</thead>
<tbody>
<tr>
<td>Parabolic spandrel</td>
<td>$3a/4$</td>
<td>$3h/10$</td>
<td>$ah/3$</td>
</tr>
<tr>
<td>General spandrel</td>
<td>$n+1/a$</td>
<td>$n+1/hb$</td>
<td>$ah/n+1$</td>
</tr>
<tr>
<td>Circular sector</td>
<td>$2r\sin\alpha/3\pi$</td>
<td>0</td>
<td>$\alpha r^2$</td>
</tr>
</tbody>
</table>

Fig. 5.8A  Centroids of common shapes of areas.