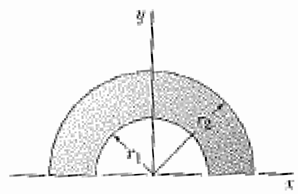
# Homework #8 Solution



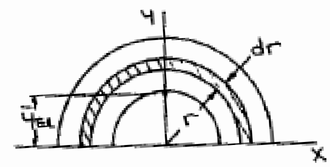

**PROBLEM 5.39**

Determine by direct integration the centroid of the area shown.

**SOLUTION**

First note that symmetry implies $\bar{x} = 0$ ◀

For the element (EL) shown,




$$\bar{y}_{EL} = \frac{2r}{\pi} \quad \text{(Figure 5.8B)}$$

$$dA = \pi r dr$$

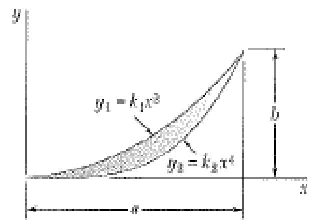
Then $$A = \int dA = \int_{r_1}^{r_2} \pi r dr = \pi\left(\frac{r^2}{2}\right)\Bigg|_{r_1}^{r_2} = \frac{\pi}{2}\left(r_2^2 - r_1^2\right)$$

and $$\int \bar{y}_{EL} dA = \int_{r_1}^{r_2} \frac{2r}{\pi}(\pi r dr) = 2\left(\frac{1}{3}r^3\right)\Bigg|_{r_1}^{r_2} = \frac{2}{3}\left(r_2^3 - r_1^3\right)$$

So $$\bar{y}A = \int \bar{y}_{EL} dA: \quad \bar{y}\left[\frac{\pi}{2}\left(r_2^2 - r_1^2\right)\right] = \frac{2}{3}\left(r_2^3 - r_1^3\right)$$

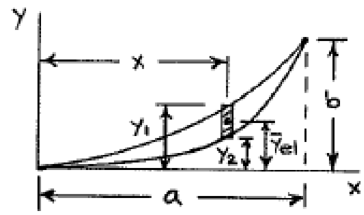
or $\bar{y} = \dfrac{4}{3\pi}\dfrac{r_2^3 - r_1^3}{r_2^2 - r_1^2}$ ◀

## PROBLEM 5.40




Determine by direct integration the centroid of the area shown. Express your answer in terms of $a$ and $b$.

## SOLUTION




$$y_1 = k_1x^2 \quad \text{but} \quad b = k_1a^2 \quad y_1 = \frac{b}{a^2}x^2$$

$$y_2 = k_2x^4 \quad \text{but} \quad b = k_2a^4 \quad y_2 = \frac{b}{a^4}x^4$$

$$dA = (y_2 - y_1)dx = \frac{b}{a^2}\left(x^2 - \frac{x^4}{a^2}\right)dx$$

$$\overline{x}_{EL} = x$$

$$\overline{y}_{EL} = \frac{1}{2}(y_1 + y_2)$$

$$= \frac{b}{2a^2}\left(x^2 + \frac{x^4}{a^2}\right)$$

$$A = \int dA = \frac{b}{a^2}\int_0^a\left(x^2 - \frac{x^4}{a^2}\right)dx$$

$$= \frac{b}{a^2}\left[\frac{x^3}{3} - \frac{x^5}{5a^2}\right]_0^a$$

$$= \frac{2}{15}ba$$

$$\int \overline{x}_{EL}dA = \int_0^a x\frac{b}{a^2}\left(x^2 - \frac{x^4}{a^2}\right)dx$$

$$= \frac{b}{a^2}\int_0^a\left(x^3 - \frac{x^5}{a^2}\right)dx$$

$$= \frac{b}{a^2}\left[\frac{x^4}{4} - \frac{x^6}{6a^2}\right]_0^a$$

$$= \frac{1}{12}a^2b$$

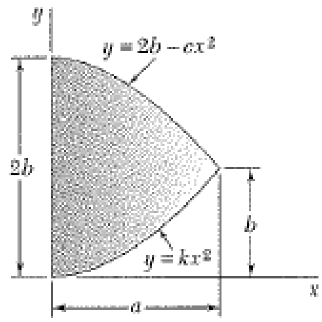
$$\int \overline{y}_{EL}dA = \int_0^a \frac{b}{2a^2}\left(x^2 + \frac{x^4}{a^2}\right)\frac{b}{a^2}\left(x^2 - \frac{x^4}{a^2}\right)dx$$

$$= \frac{b^2}{2a^4}\int_0^a\left(x^4 - \frac{x^8}{a^4}\right)dx$$

$$= \frac{b^2}{2a^4}\left[\frac{x^5}{5} - \frac{x^9}{9a^4}\right]_0^a = \frac{2}{45}ab^2$$

$$\overline{x}A = \int \overline{x}_{EL}dA: \quad \overline{x}\left(\frac{2}{15}ba\right) = \frac{1}{12}a^2b \qquad \overline{x} = \frac{5}{8}a \blacktriangleleft$$

$$\overline{y}A = \int \overline{y}_{EL}dA: \quad \overline{y}\left(\frac{2}{15}ba\right) = \frac{2}{45}ab^2 \qquad \overline{y} = \frac{1}{3}b \blacktriangleleft$$

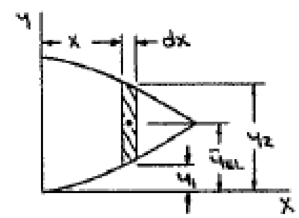



## PROBLEM 5.41

Determine by direct integration the centroid of the area shown. Express your answer in terms of $a$ and $b$.

## SOLUTION

First note that symmetry implies $\bar{y} = b$ ◀




At $x = a, \quad y = b$

$$y_1: \quad b = ka^2 \quad \text{or} \quad k = \frac{b}{a^2}$$

Then $y_1 = \frac{b}{a^2}x^2$

$$y_2: \quad b = 2b - ca^2$$

or $c = \frac{b}{a^2}$

Then $y_2 = b\left(2 - \frac{x^2}{a^2}\right)$

Now
$$dA = (y_2 - y_1)dx_2 = \left[b\left(2 - \frac{x^2}{a^2}\right) - \frac{b}{a^2}x^2\right]dx$$
$$= 2b\left(1 - \frac{x^2}{a^2}\right)dx$$

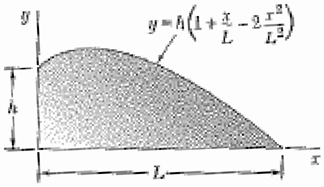
and $\bar{x}_{EL} = x$

Then
$$A = \int dA \int_0^a 2b\left(1 - \frac{x^2}{a^2}\right)dx = 2b\left[x - \frac{x^3}{3a^2}\right]_0^a = \frac{4}{3}ab$$

and
$$\int \bar{x}_{EL}dA = \int_0^a x\left[2b\left(1 - \frac{x^2}{a^2}\right)dx\right] = 2b\left[\frac{x^2}{2} - \frac{x^4}{4a^2}\right]_0^a = \frac{1}{2}a^2b$$

$$\bar{x}A = \int \bar{x}_{EL}dA: \quad \bar{x}\left(\frac{4}{3}ab\right) = \frac{1}{2}a^2b$$

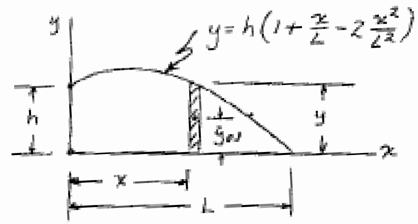
$\bar{x} = \frac{3}{8}a$ ◀




## PROBLEM 5.42

Determine by direct integration the centroid of the area shown.

## SOLUTION




$$\bar{x}_{EL} = x \quad \bar{y}_{EL} = \frac{1}{2}y \quad dA = y\,dx$$

$$A = \int dA = \int_0^L h\left(1 + \frac{x}{L} - 2\frac{x^2}{L^2}\right)dx = h\left[x + \frac{x^2}{2L} - \frac{2}{3}\frac{x^3}{L^2}\right]_0^L = \frac{5}{6}hL$$

$$\int \bar{x}_{EL}\,dA = \int_0^L xh\left(1 + \frac{x}{L} - 2\frac{x^2}{L^2}\right)dx = h\int_0^L\left(x + \frac{x^2}{L} - 2\frac{x^3}{L^2}\right)dx$$

$$= h\left[\frac{x^2}{2} + \frac{1}{3}\frac{x^3}{L} - \frac{2}{4}\frac{x^4}{L^2}\right]_0^L = \frac{1}{3}hL^2$$

$$\bar{x}A = \int \bar{x}_{EL}\,dA: \quad \bar{x}\left(\frac{5}{6}hL\right) = \frac{1}{3}hL^2$$

$$\bar{x} = \frac{2}{5}L \blacktriangleleft$$

$$A = \frac{5}{6}hL \quad \bar{y}_{EL} = \frac{1}{2}y \quad y = h\left(1 + \frac{x}{L} - 2\frac{x^2}{L^2}\right)$$

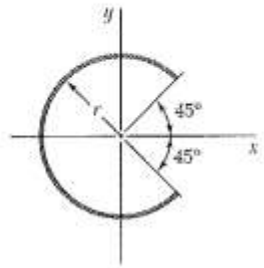
$$\int \bar{y}_{EL}\,dA = \frac{1}{2}\int y^2\,dx = \frac{h^2}{2}\int_0^L\left(1 + \frac{x}{L} - 2\frac{x^2}{L^2}\right)^2 dx$$

$$= \frac{h^2}{2}\int_0^L\left(1 + \frac{x^2}{L^2} + 4\frac{x^4}{L^4} + 2\frac{x}{L} - 4\frac{x^2}{L^2} - 4\frac{x^3}{L^3}\right)dx$$

$$= \frac{h^2}{2}\left[x + \frac{x^3}{3L^2} + \frac{4x^5}{5L^4} + \frac{x^2}{L} - \frac{4x^3}{3L^2} - \frac{x^4}{L^3}\right]_0^L = \frac{4}{10}h^2L$$

$$\bar{y}A = \int \bar{y}_{EL}\,dA: \quad \bar{y}\left(\frac{5}{6}hL\right) = \frac{4}{10}h^2L$$

$$\bar{y} = \frac{12}{25}h \blacktriangleleft$$

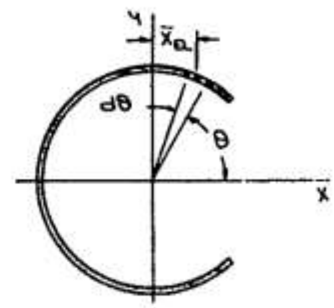



## PROBLEM 5.45

A homogeneous wire is bent into the shape shown. Determine by direct integration the $x$ coordinate of its centroid.

## SOLUTION

First note that because the wire is homogeneous, its center of gravity coincides with the centroid of the corresponding line.




Now $\qquad \bar{x}_{EL} = r\cos\theta \quad \text{and} \quad dL = r\,d\theta$

Then $\qquad L = \int dL = \int_{\pi/4}^{7\pi/4} r\,d\theta = r[\theta]_{\pi/4}^{7\pi/4} = \frac{3}{2}\pi r$

and

$$\int \bar{x}_{EL}\,dL = \int_{\pi/4}^{7\pi/4} r\cos\theta(r\,d\theta)$$

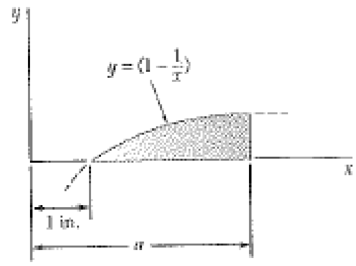
$$= r^2[\sin\theta]_{\pi/4}^{7\pi/4}$$

$$= r^2\left(-\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}\right)$$

$$= -r^2\sqrt{2}$$

Thus $\qquad \bar{x}L = \int \bar{x}\,dL: \quad \bar{x}\left(\frac{3}{2}\pi r\right) = -r^2\sqrt{2}$

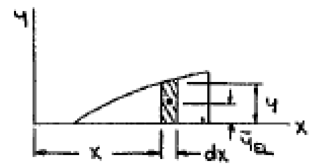
$$\bar{x} = -\frac{2\sqrt{2}}{3\pi} r \quad ◀$$




## PROBLEM 5.50

Determine the centroid of the area shown when $a = 2$ in.

## SOLUTION




We have $\bar{x}_{EL} = x$

$$\bar{y}_{EL} = \frac{1}{2}y = \frac{1}{2}\left(1 - \frac{1}{x}\right)$$

and $dA = y\,dx = \left(1 - \frac{1}{x}\right)dx$

Then $$A = \int dA = \int_1^a \left(1 - \frac{1}{x}\right)\frac{dx}{2} = [x - \ln x]_1^a = (a - \ln a - 1)\text{ in}^2$$

and $$\int \bar{x}_{EL}\,dA = \int_1^a x\left[\left(1 - \frac{1}{x}\right)dx\right] = \left[\frac{x^2}{2} - x\right]_1^a = \left(\frac{a^2}{2} - a + \frac{1}{2}\right)\text{in}^3$$

$$\int \bar{y}_{EL}\,dA = \int_1^a \frac{1}{2}\left(1 - \frac{1}{x}\right)\left[\left(1 - \frac{1}{x}\right)dx\right] = \frac{1}{2}\int_1^a \left(1 - \frac{2}{x} + \frac{1}{x^2}\right)dx$$

$$= \frac{1}{2}\left[x - 2\ln x - \frac{1}{x}\right]_1^a = \frac{1}{2}\left(a - 2\ln a - \frac{1}{a}\right)\text{in}^3$$

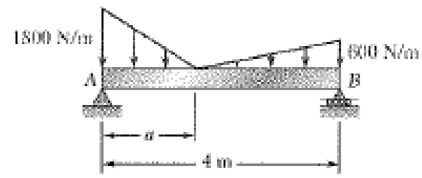
$$\bar{x}A = \int \bar{x}_{EL}\,dA: \quad \bar{x} = \frac{\frac{a^2}{2} - a + \frac{1}{2}}{a - \ln a - 1}\text{ in.}$$

$$\bar{y}A = \int \bar{y}_{EL}\,dA: \quad \bar{y} = \frac{a - 2\ln a - \frac{1}{a}}{2(a - \ln a - 1)}\text{ in.}$$

Find $\bar{x}$ and $\bar{y}$ when $a = 2$ in.

We have $$\bar{x} = \frac{\frac{1}{2}(2)^2 - 2 + \frac{1}{2}}{2 - \ln 2 - 1}$$ or $\bar{x} = 1.629$ in. ◀

and $$\bar{y} = \frac{2 - 2\ln 2 - \frac{1}{2}}{2(2 - \ln 2 - 1)}$$ or $\bar{y} = 0.1853$ in. ◀

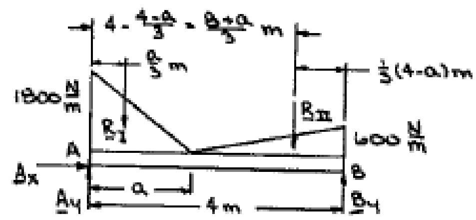



## PROBLEM 5.77

Determine (*a*) the distance *a* so that the reaction at support *B* is minimum, (*b*) the corresponding reactions at the supports.

## SOLUTION

(*a*)



We have $R_I = \frac{1}{2}(a\text{ m})(1800\text{ N/m}) = 900a\text{ N}$

$$R_{II} = \frac{1}{2}[(4-a)\text{m}](600\text{ N/m}) = 300(4-a)\text{ N}$$

Then $+\circlearrowleft \Sigma M_A = 0$: $-\left(\frac{a}{3}\text{m}\right)(900a\text{ N}) - \left(\frac{8+a}{3}\text{ m}\right)[300(4-a)\text{N}] + (4\text{ m})B_y = 0$

or $$B_y = 50a^2 - 100a + 800 \qquad (1)$$

Then $\frac{dB_y}{da} = 100a - 100 = 0$ or $a = 1.000\text{ m}$ ◀

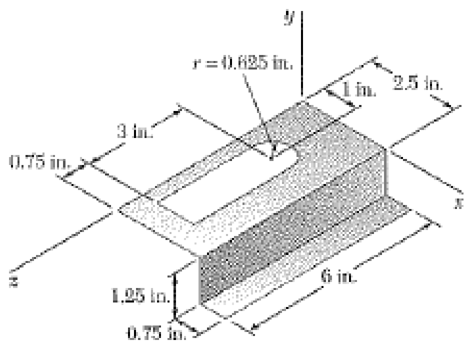
(*b*) From Eq. (1): $B_y = 50(1)^2 - 100(1) + 800 = 750\text{ N}$ $\mathbf{B} = 750\text{ N}\uparrow$ ◀

and $\xrightarrow{+} \Sigma F_x = 0$: $A_x = 0$

$+\uparrow \Sigma F_y = 0$: $A_y - 900(1)\text{ N} - 300(4-1)\text{ N} + 750\text{ N} = 0$

or $A_y = 1050\text{ N}$ $\mathbf{A} = 1050\text{ N}\uparrow$ ◀

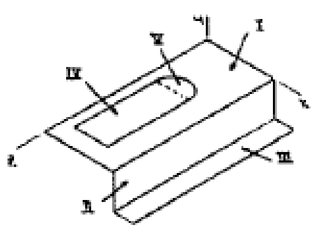
**PROBLEM 5.111**




A mounting bracket for electronic components is formed from sheet metal of uniform thickness. Locate the center of gravity of the bracket.

**SOLUTION**

First, assume that the sheet metal is homogeneous so that the center of gravity of the bracket coincides with the centroid of the corresponding area. Then (see diagram)



$$\bar{z}_V = 2.25 - \frac{4(0.625)}{3\pi} = 1.98474 \text{ in.}$$

$$A_V = -\frac{\pi}{2}(0.625)^2 = -0.61359 \text{ in}^2$$

| | $A$, in$^2$ | $\bar{x}$, in. | $\bar{y}$, in. | $\bar{z}$, in. | $\bar{x}A$, in$^3$ | $\bar{y}A$, in$^3$ | $\bar{z}A$, in$^3$ |
|---|---|---|---|---|---|---|---|
| I | $(2.5)(6) = 15$ | 1.25 | 0 | 3 | 18.75 | 0 | 45 |
| II | $(1.25)(6) = 7.5$ | 2.5 | −0.625 | 3 | 18.75 | −4.6875 | 22.5 |
| III | $(0.75)(6) = 4.5$ | 2.875 | −1.25 | 3 | 12.9375 | −5.625 | 13.5 |
| IV | $-\left(\frac{5}{4}\right)(3) = -3.75$ | 1.0 | 0 | 3.75 | 3.75 | 0 | −14.0625 |
| V | −0.61359 | 1.0 | 0 | 1.9847 | 0.61359 | 0 | −1.21782 |
| Σ | 22.6364 | | | | 46.0739 | 10.3125 | 65.7197 |

We have

$$\bar{X}\Sigma A = \Sigma \bar{x}A$$

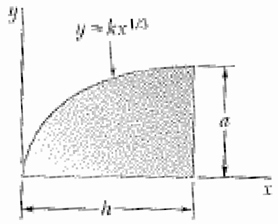
$$\bar{X}(22.6364 \text{ in}^2) = 46.0739 \text{ in}^3 \qquad \text{or} \qquad \bar{X} = 2.04 \text{ in.} \blacktriangleleft$$

$$\bar{Y}\Sigma A = \Sigma \bar{y}A$$

$$\bar{Y}(22.6364 \text{ in}^2) = -10.3125 \text{ in}^3 \qquad \text{or} \qquad \bar{Y} = -0.456 \text{ in.} \blacktriangleleft$$

$$\bar{Z}\Sigma A = \Sigma \bar{z}A$$

$$\bar{Z}(22.6364 \text{ in}^2) = 65.7197 \text{ in}^3 \qquad \text{or} \qquad \bar{Z} = 2.90 \text{ in.} \blacktriangleleft$$




## PROBLEM 5.125

Locate the centroid of the volume obtained by rotating the shaded area about the $x$-axis.

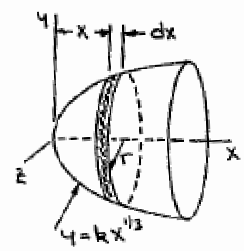
## SOLUTION

First note that symmetry implies $\bar{y} = 0$ ◀

$\bar{z} = 0$ ◀

Choose as the element of volume a disk of radius $r$ and thickness $dx$. Then

$$dV = \pi r^2 dx, \quad x_{EL} = x$$




Now $r = kx^{1/3}$

so that $dV = \pi k^2 x^{2/3} dx$

at $x = h,\ y = a,$ $a = kh^{1/3}$

or $k = \dfrac{a}{h^{1/3}}$

Then $dV = \pi \dfrac{a^2}{h^{2/3}} x^{2/3} dx$

and

$$V = \int_0^h \pi \frac{a^2}{h^{2/3}} x^{2/3} dx$$

$$= \pi \frac{a^2}{h^{2/3}} \left[ \frac{3}{5} x^{5/3} \right]_0^h$$

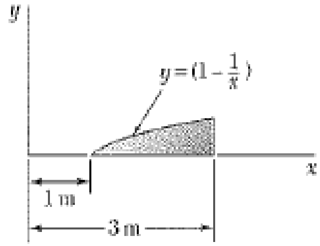
$$= \frac{3}{5} \pi a^2 h$$

Also

$$\int \bar{x}_{EL} dV = \int_0^h x \left( \pi \frac{a^2}{h^{2/3}} x^{2/3} dx \right) = \pi \frac{a^2}{h^{2/3}} \left[ \frac{3}{8} x^{8/3} \right]_0^h$$

$$= \frac{3}{8} \pi a^2 h^2$$

Now $\bar{x}V = \int \bar{x} dV: \quad \bar{x}\left( \dfrac{3}{5} \pi a^2 h \right) = \dfrac{3}{8} \pi a^2 h^2$ or $\bar{x} = \dfrac{5}{8} h$ ◀




## PROBLEM 5.126

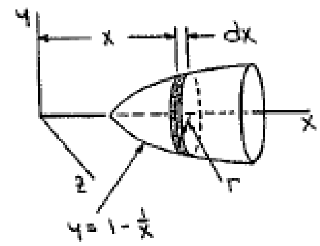
Locate the centroid of the volume obtained by rotating the shaded area about the $x$-axis.

## SOLUTION

First, note that symmetry implies $\overline{y}=0$ ◀

$\overline{z}=0$ ◀

Choose as the element of volume a disk of radius $r$ and thickness $dx$.




Then $$dV=\pi r^2 dx, \quad \overline{x}_{EL}=x$$

Now $r=1-\dfrac{1}{x}$ so that $$dV=\pi\left(1-\frac{1}{x}\right)^2 dx$$

$$=\pi\left(1-\frac{2}{x}+\frac{1}{x^2}\right)dx$$

Then $$V=\int_1^3 \pi\left(1-\frac{2}{x}+\frac{1}{x^2}\right)dx=\pi\left[x-2\ln x-\frac{1}{x}\right]_1^3$$

$$=\pi\left[\left(3-2\ln 3-\frac{1}{3}\right)-\left(1-2\ln 1-\frac{1}{1}\right)\right]$$

$$=(0.46944\pi)\text{ m}^3$$

and $$\int \overline{x}_{EL}dV=\int_1^3 x\left[\pi\left(1-\frac{2}{x}+\frac{1}{x^2}\right)dx\right]=\pi\left[\frac{x^2}{2}-2x+\ln x\right]_1^3$$

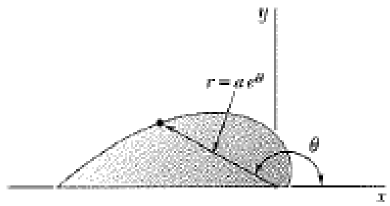
$$=\pi\left\{\left[\frac{3^2}{2}-2(3)+\ln 3\right]-\left[\frac{1^3}{2}-2(1)+\ln 1\right]\right\}$$

$$=(1.09861\pi)\text{ m}$$

Now $$\overline{x}V=\int \overline{x}_{EL}dV: \quad \overline{x}(0.46944\pi\text{ m}^3)=1.09861\pi\text{ m}^4$$

or $\overline{x}=2.34$ m ◀

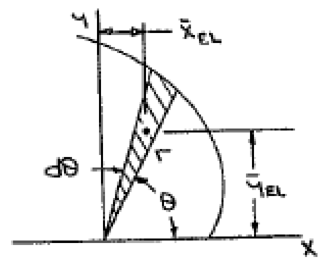
**Bonus**




**PROBLEM 5.49***

Determine by direct integration the centroid of the area shown.

**SOLUTION**




We have
$$\bar{x}_{EL} = \frac{2}{3} r \cos\theta = \frac{2}{3} a e^{\theta} \cos\theta$$
$$\bar{y}_{EL} = \frac{2}{3} r \sin\theta = \frac{2}{3} a e^{\theta} \sin\theta$$

and
$$dA = \frac{1}{2}(r)(r d\theta) = \frac{1}{2} a^2 e^{2\theta} d\theta$$

Then
$$A = \int dA = \int_0^{\pi} \frac{1}{2} a^2 e^{2\theta} d\theta = \frac{1}{2} a^2 \left[\frac{1}{2} e^{2\theta}\right]_0^{\pi}$$
$$= \frac{1}{4} a^2 (e^{2\pi} - 1)$$
$$= 133.623 a^2$$

and
$$\int \bar{x}_{EL} dA = \int_0^{\pi} \frac{2}{3} a e^{\theta} \cos\theta \left(\frac{1}{2} a^2 e^{2\theta} d\theta\right)$$
$$= \frac{1}{3} a^3 \int_0^{\pi} e^{3\theta} \cos\theta d\theta$$

To proceed, use integration by parts, with
$$u = e^{3\theta} \quad \text{and} \quad du = 3e^{3\theta} d\theta$$
$$dv = \cos\theta d\theta \quad \text{and} \quad v = \sin\theta$$

Then
$$\int e^{3\theta} \cos\theta d\theta = e^{3\theta} \sin\theta - \int \sin\theta (3e^{3\theta} d\theta)$$

Now let
$$u = e^{3\theta} \quad \text{then} \quad du = 3e^{3\theta} d\theta$$
$$dv = \sin\theta d\theta, \quad \text{then} \quad v = -\cos\theta$$

Then
$$\int e^{3\theta} \cos\theta d\theta = e^{3\theta} \sin\theta - 3\left[-e^{3\theta} \cos\theta - \int (-\cos\theta)(3e^{3\theta} d\theta)\right]$$

## PROBLEM 5.49* (Continued)

so that
$$\int e^{3\theta}\cos\theta d\theta = \frac{e^{3\theta}}{10}(\sin\theta + 3\cos\theta)$$

$$\int x_{EL}dA = \frac{1}{3}a^3\left[\frac{e^{3\theta}}{10}(\sin\theta + 3\cos\theta)\right]_0^\pi$$

$$= \frac{a^3}{30}(-3e^{3\pi} - 3) = -1239.26a^3$$

Also,
$$\int \overline{y}_{EL}dA = \int_0^\pi \frac{2}{3}ae^\theta \sin\theta\left(\frac{1}{2}a^2e^{2\theta}d\theta\right)$$

$$= \frac{1}{3}a^3\int_0^\pi e^{3\theta}\sin\theta d\theta$$

Use integration by parts, as above, with

$$u = e^{3\theta} \quad \text{and} \quad du = 3e^{3\theta}d\theta$$

$$dv = \int \sin\theta d\theta \quad \text{and} \quad v = -\cos\theta$$

Then
$$\int e^{3\theta}\sin\theta d\theta = -e^{3\theta}\cos\theta - \int(-\cos\theta)(3e^{3\theta}d\theta)$$

so that
$$\int e^{3\theta}\sin\theta d\theta = \frac{e^{3\theta}}{10}(-\cos\theta + 3\sin\theta)$$

$$\int \overline{y}_{EL}dA = \frac{1}{3}a^3\left[\frac{e^{3\theta}}{10}(-\cos\theta + 3\sin\theta)\right]_0^\pi$$

$$= \frac{a^3}{30}(e^{3\pi} + 1) = 413.09a^3$$

Hence,
$$\overline{x}A = \int x_{EL}dA: \quad \overline{x}(133.623a^2) = -1239.26a^3 \qquad \text{or } \overline{x} = -9.27a \blacktriangleleft$$

$$\overline{y}A = \int \overline{y}_{EL}dA: \quad \overline{y}(133.623a^2) = 413.09a^3 \qquad \text{or } \overline{y} = 3.09a \blacktriangleleft$$