Question 1 (20 + 5 points)

(a) (10 points) In each of the two following examples, in the column to the right, draw a free body diagram (FBD) of the body to be isolated, shown in the middle column. Dimensions and numerical values are omitted for simplicity.

<table>
<thead>
<tr>
<th>Description</th>
<th>Body to be Isolated</th>
<th>FBD</th>
</tr>
</thead>
<tbody>
<tr>
<td>Uniform pole of mass $m$ hoisted into position by winch. Horizontal supporting surface notched to prevent slipping of the pole.</td>
<td><img src="image1" alt="Uniform pole" /></td>
<td><img src="image2" alt="FBD" /></td>
</tr>
<tr>
<td>Uniform horizontal bar of mass $m$ suspended by vertical cable at $A$ and supported by a smooth surface at $B$</td>
<td><img src="image3" alt="Uniform horizontal bar" /></td>
<td><img src="image4" alt="FBD" /></td>
</tr>
<tr>
<td>Uniform grooved wheel of mass $m$ supported by a rough surface and by action of a horizontal cable</td>
<td><img src="image5" alt="Uniform grooved wheel" /></td>
<td><img src="image6" alt="FBD" /></td>
</tr>
<tr>
<td>Uniform heavy plate of mass $m$ supported in vertical plane by cable $C$ and hinge $A$</td>
<td><img src="image7" alt="Uniform heavy plate" /></td>
<td><img src="image8" alt="FBD" /></td>
</tr>
</tbody>
</table>
(b) (15 points) In the following 5 sketches determine whether (a) the body is completely, partially or improperly constrained, (b) the reactions are statically determinate or indeterminate, (c) the equilibrium of the body is maintained in the position shown. All connections in cases 1-4 consist of frictionless pins, rollers or short links. In case 5 the surfaces are smooth. Explain your answers. Draw any sketches/FBDs needed.

Solution:

1. 4 reactions with concurrent lines of actions – improper constraint, statically indeterminate, equilibrium not maintained.
2. 2 vertical reactions – partially constrained, statically determinate, equilibrium maintained.
3. 4 reactions with non-concurrent or parallel lines of action – properly constrained, statically indeterminate, equilibrium maintained.
4. 3 reactions with non-concurrent or parallel lines of action – properly constrained, statically determinate, equilibrium maintained.
5. 3 reactions with concurrent lines of action – improperly constrained, statically indeterminate, equilibrium not maintained.
**Question 2 (20 points)** A Pratt roof truss is loaded as shown in the figure to the right.

(a) (10 points) Neglect any horizontal reactions at the supports and solve for the forces in all the members if \( P = Q = 1 \) kN.

**Solution:**

By inspection of joint B (special case), \( BA=BC=3.35 \) kN C, \( BH=P=1 \) kN C.
(b) (5 points) If $P = 0$ kN, and $Q = 4$ kN, what are the forces in members $AB$ and $AH$? Explain.

Solution:

The symmetry no longer exists. However, equilibrium of moments around $E$ gives the reaction at $A$, which is the same as in part (a). Under these conditions, the analysis of joint $A$ remains the same, and the forces in $AB$ and $AH$ are the same as in (b).

(e) (5 points) Under the loading condition in part (b) identify by inspection those members in which the forces are zero. Explain.

Solution:

With load $P=0$, joint $B$ is a special case, giving $BH=0$. Given that, joints $H$ and $G$ are also special cases, giving $HC=GC=0$

Question 3 (30 points)

(a) (15 points) Determine the coordinates of the centroid of the bracket in the figure to the right (use the attached tables of centroids of common bodies).

Solution:
(b) (10 points) Determine by integration the \( y \) coordinate of the centroid of the shaded area in the figure to the right. Solutions by other methods will not carry any credit!

Solution:

\[
\begin{align*}
\int y_c \, dA &= \int_0^a \left( \frac{y_1 + y_2}{2} \right) (y_2 - y_1) \, dx = \frac{1}{2} \int_0^a (y_2^2 - y_1^2) \, dx \\
&= \frac{1}{2} \int_0^a \left[ b^2 \left(1 + \frac{x}{a}\right)^2 - \frac{b^2}{a^2} x \right] \, dx \\
&= \frac{1}{2} \left[ b^2 \left( x + \frac{x^2}{2a} + \frac{x^3}{3a^2} \right) - \frac{b^2}{a} x^2 \right]_0^a = \frac{11}{12} ab^2 \\
\bar{y} &= \frac{\int y_c \, dA}{A} = \frac{11ab^2/12}{5a^2b/6} = \frac{11}{10} b
\end{align*}
\]

(e) (5 points) By treating the shaded area in part (b) as a composite body, find the \( x \) coordinate of the centroid. Use the attached tables of centroids of common bodies.

<table>
<thead>
<tr>
<th></th>
<th>( A )</th>
<th>( \bar{x} )</th>
<th>( A \bar{x} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>( \frac{1}{2} * ab )</td>
<td>( \frac{2}{3} * a )</td>
<td>( \frac{1}{3} * a^2b )</td>
</tr>
<tr>
<td>2</td>
<td>( \frac{1}{3} * ab )</td>
<td>( \frac{3}{10} * a )</td>
<td>( \frac{1}{10} * a^2b )</td>
</tr>
<tr>
<td>3</td>
<td>( \frac{5}{6} * ab )</td>
<td></td>
<td>( \frac{13}{30} * a^2b )</td>
</tr>
</tbody>
</table>

\[
\bar{X} = \sum_i \bar{x}_i A_i \Rightarrow \bar{X} = \frac{\sum_i \bar{x}_i A_i}{\sum_i A_i}
\]

\[
\bar{X} = \frac{13/30 * a^2b}{5/6 * ab} = \frac{13}{25} * a
\]
Question 3 (30 + 10 points) The 20 X 20-in square plate weighs 56 lb and is supported by three vertical wires as shown in the figure to the right.

(a) (13 points) Determine the tension in each wire.

Solution:

The center of mass for the application of \( W \) is at the center of the plate.

Equilibrium of moments around \( z \) axis passing through \( B \) and \( C \)

\[
\sum M_{Bz} = 0 = W \times 6 \text{ in} - T_A \times 16 \text{ in} \quad \Rightarrow \quad T_A = \frac{56 \text{ lb} \times 6 \text{ in}}{16 \text{ in}} = 21 \text{ lb}
\]

Because of symmetry, \( T_C = T_B = T \).

Equilibrium of forces in the \( y \) direction:

\[
\sum F_y = 0 = 2T + T_A - W \quad \Rightarrow \quad T = \frac{56 \text{ lb} - 21 \text{ lb}}{2} = 17.5 \text{ lb}
\]

Answer:

\( T_A = 21 \text{ lb} \)

\( T_B = T_C = 17.5 \text{ lb} \)
(b) (12 points) If a vertical force of 10 lb is applied down on the plate, what is the point of application if it results in all tensions being equal (8 points)? What is the corresponding tension (4)?

Solution:

Now, \( W_b = 10 \text{ lb} \) is applied to the plate at a point \( (x_b, z_b) \). Since all the tensions are equal, from an equilibrium of forces in the \( y \) direction:

\[
\sum F_y = 0 = 3T - W - W_b \quad \Rightarrow \quad T = \frac{56 \text{ lb} + 10 \text{ lb}}{3} = 22 \text{ lb}
\]

From symmetry, to maintain an equilibrium of moments around the \( x \) axis that passes through \( A \), and because \( T_B = T_C \), it follows that \( z_b = 10 \text{ in} \).

To calculate the \( x_b \) consider equilibrium of moments around the \( z \) axis passing through \( B \).

\[
\sum M_{Bz} = 0 = W_b \ast (16 \text{ in} - x_b) + W \ast 6 \text{ in} - T \ast 16 \text{ in}
\]

\[
(16 \text{ in} - x_b) = \frac{22 \text{ lb} \ast 16 \text{ in} - 56 \text{ lb} \ast 6 \text{ in}}{10 \text{ lb}} = 1.6 \text{ in}
\]

\[
x_b = 16 \text{ in} - 1.6 \text{ in} = 14.4 \text{ in}
\]

(c) (5 points) What is the maximum distance \( d \) from the \( z \) axis for a 150 lb force down to be applied on the plate without the plate tipping?

Solution:

For a plate to tip means in this case to start rotating around the \( BC \) axis. The extreme case, just before tipping means that \( T_A = 0 \). From equilibrium of moments around \( BC \) gives us:

\[
\sum M_{BC} = 0 = W \ast 6 \text{ in} - W_b \ast (d - 16) \text{ in} \quad \Rightarrow \quad d = \frac{(56 \text{ lb} \ast 6 \text{ in})}{150 \text{ lb}} + 16 \text{ in} = 18.24 \text{ in}
\]
(d) **Bonus** (10 points) Show on a sketch the area of the plate over which the force in part (c) can act without the plate tipping. Mark relevant distances.

**Solution:**

The plate can tip around one of the three axes $BC$, $AB$ and $AC$. In each of the extreme cases, just before tipping, the tension in the corresponding third cable is zero. In the previous part we calculate the distance $d$ for the case of tipping around the $BC$ axis.

The area that the force can be applied to without tipping the plate is marked in gray in the top view of the plate:

$$\tan \alpha = \frac{10}{16} \Rightarrow \alpha = 32^\circ$$

$$d'' = 10 \text{ in} \cdot \sin \alpha = 5.3 \text{ in}$$

$$\sum M_{AB} = 0 = W_b \cdot d' - W \cdot d''$$

$$d' = \frac{W \cdot d''}{W_b} = \frac{56 \text{ lb} \cdot 5.3 \text{ in}}{150 \text{ lb}} = 1.98 \text{ in}$$

$$(16 - x') = \frac{d'}{\sin \alpha} \Rightarrow x' = 16 \text{ in} - \frac{d'}{\sin \alpha} = 12.27 \text{ in}$$

$$z' = x' \cdot \tan \alpha = 7.67 \text{ in}$$

$$20 - z' = 12.33 \text{ in}$$

**Show your work!**

**Good Luck!**
## Centroids of Common 1D Bodies

<table>
<thead>
<tr>
<th>Shape</th>
<th>$\bar{x}$</th>
<th>$\bar{y}$</th>
<th>Length</th>
</tr>
</thead>
<tbody>
<tr>
<td>Quarter-circular arc</td>
<td>$\frac{2r}{\pi}$</td>
<td>$\frac{2r}{\pi}$</td>
<td>$\frac{\pi r}{2}$</td>
</tr>
<tr>
<td>Semicircular arc</td>
<td>0</td>
<td>$\frac{2r}{\pi}$</td>
<td>$\pi r$</td>
</tr>
<tr>
<td>Arc of circle</td>
<td>$\frac{r \sin \alpha}{\alpha}$</td>
<td>0</td>
<td>$2\alpha r$</td>
</tr>
</tbody>
</table>

*Fig. 5.8B* Centroids of common shapes of lines.

## Centroids of Common 2D Bodies

<table>
<thead>
<tr>
<th>Shape</th>
<th>$\bar{x}$</th>
<th>$\bar{y}$</th>
<th>Area</th>
</tr>
</thead>
<tbody>
<tr>
<td>Triangular area</td>
<td>$\frac{h}{3}$</td>
<td>$\frac{bh}{2}$</td>
<td></td>
</tr>
<tr>
<td>Quarter-circular area</td>
<td>$\frac{4r}{3\pi}$</td>
<td>$\frac{4r}{3\pi}$</td>
<td>$\frac{\pi r^2}{4}$</td>
</tr>
<tr>
<td>Semicircular area</td>
<td>0</td>
<td>$\frac{4r}{3\pi}$</td>
<td>$\frac{\pi r^2}{2}$</td>
</tr>
<tr>
<td>Quarter-elliptical area</td>
<td>$\frac{4a}{3\pi}$</td>
<td>$\frac{4b}{3\pi}$</td>
<td>$\frac{\pi ab}{4}$</td>
</tr>
<tr>
<td>Semielliptical area</td>
<td>0</td>
<td>$\frac{4b}{3\pi}$</td>
<td>$\frac{\pi ab}{2}$</td>
</tr>
<tr>
<td>Semiparabolic area</td>
<td>$\frac{3a}{8}$</td>
<td>$\frac{3h}{5}$</td>
<td>$\frac{2ah}{3}$</td>
</tr>
<tr>
<td>Parabolic area</td>
<td>0</td>
<td>$\frac{3h}{5}$</td>
<td>$\frac{4ah}{3}$</td>
</tr>
</tbody>
</table>
### Centroids of Common 3D Bodies

<table>
<thead>
<tr>
<th>Shape</th>
<th>$\bar{x}$</th>
<th>Volume</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Hemisphere</strong></td>
<td>$\frac{3a}{8}$</td>
<td>$\frac{2}{3} \pi a^3$</td>
</tr>
<tr>
<td><strong>Semiellipsoid of revolution</strong></td>
<td>$\frac{3h}{8}$</td>
<td>$\frac{2}{3} \pi a^2 h$</td>
</tr>
</tbody>
</table>

**Fig. 5.8A** Centroids of common shapes of areas.
<table>
<thead>
<tr>
<th>Shape</th>
<th>$h$</th>
<th>$\frac{1}{2} \pi a^2 h$</th>
<th>$\frac{h}{3}$</th>
<th>$\frac{1}{3} \pi a^2 h$</th>
<th>$\frac{h}{4}$</th>
<th>$\frac{1}{3} abh$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Paraboloid of revolution</td>
<td></td>
<td></td>
<td>$\frac{h}{3}$</td>
<td>$\frac{1}{2} \pi a^2 h$</td>
<td>$\frac{h}{4}$</td>
<td>$\frac{1}{3} abh$</td>
</tr>
<tr>
<td>Cone</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Pyramid</td>
<td></td>
<td></td>
<td></td>
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<td></td>
</tr>
</tbody>
</table>

**Fig. 5.21** Centroids of common shapes and volumes.