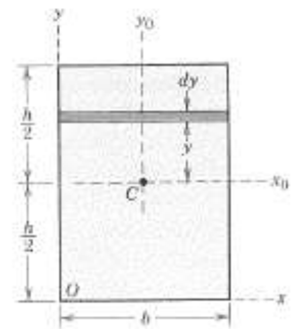
Moments of Inertia (cross sections) Problem Solutions

By integration

Example 1:

Sample Problem A/1

Determine the moments of inertia of the rectangular area about the centroidal x_0- and y_0-axes, the centroidal polar axis z_0 through C, the x-axis, and the polar axis z through O.



Solution. For the calculation of the moment of inertia $\bar{I}_x$ about the x_0-axis, a horizontal strip of area $b\,dy$ is chosen so that all elements of the strip have the same y-coordinate. Thus,

$$[I_x = \int y^2\,dA] \qquad \bar{I}_x = \int_{-h/2}^{h/2} y^2 b\,dy = \tfrac{1}{12}bh^3 \qquad \textit{Ans.}$$

By interchanging symbols the moment of inertia about the centroidal y_0-axis is

$$\bar{I}_y = \tfrac{1}{12}hb^3 \qquad \textit{Ans.}$$

The centroidal polar moment of inertia is

$$[\bar{I}_z = \bar{I}_x + \bar{I}_y] \qquad \bar{I}_z = \tfrac{1}{12}(bh^3 + hb^3) = \tfrac{1}{12}A(b^2 + h^2) \qquad \textit{Ans.}$$

By the parallel-axis theorem the moment of inertia about the x-axis is

$$[I_x = \bar{I}_x + Ad_x^2] \qquad I_x = \tfrac{1}{12}bh^3 + bh\left(\frac{h}{2}\right)^2 = \tfrac{1}{3}bh^3 = \tfrac{1}{3}Ah^2 \qquad \textit{Ans.}$$

We also obtain the polar moment of inertia about O by the parallel-axis theorem, which gives us

$$[I_z = \bar{I}_z + Ad^2] \qquad I_z = \tfrac{1}{12}A(b^2 + h^2) + A\left[\left(\frac{b}{2}\right)^2 + \left(\frac{h}{2}\right)^2\right]$$

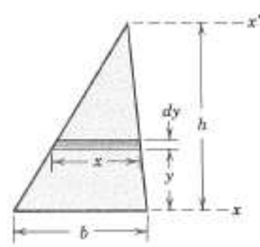
$$I_z = \tfrac{1}{3}A(b^2 + h^2) \qquad \textit{Ans.}$$

① If we had started with the second-order element $dA = dx\,dy$, integration with respect to x holding y constant amounts simply to multiplication by b and gives us the expression $y^2 b\,dy$, which we chose at the outset.

Example 2:

Sample Problem A/2

Determine the moments of inertia of the triangular area about its base and about parallel axes through its centroid and vertex.



① ② **Solution.** A strip of area parallel to the base is selected as shown in the figure, and it has the area $dA = x\,dy = [(h - y)b/h]\,dy$. By definition

$$[I_x = \int y^2\,dA] \qquad I_x = \int_0^h y^2 \frac{h-y}{h} b\,dy = b\left[\frac{y^3}{3} - \frac{y^4}{4h}\right]_0^h = \frac{bh^3}{12} \qquad \textit{Ans.}$$

By the parallel-axis theorem the moment of inertia $\bar{I}$ about an axis through the centroid, a distance $h/3$ above the x-axis, is

$$[\bar{I} = I - Ad^2] \qquad \bar{I} = \frac{bh^3}{12} - \left(\frac{bh}{2}\right)\left(\frac{h}{3}\right)^2 = \frac{bh^3}{36} \qquad \textit{Ans.}$$

A transfer from the centroidal axis to the x'-axis through the vertex gives

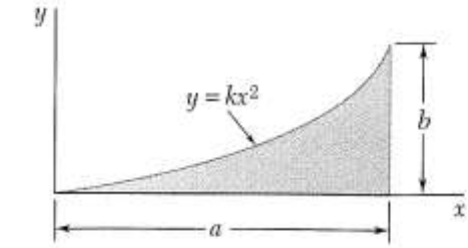
$$[I = \bar{I} + Ad^2] \qquad I_{x'} = \frac{bh^3}{36} + \left(\frac{bh}{2}\right)\left(\frac{2h}{3}\right)^2 = \frac{bh^3}{4} \qquad \textit{Ans.}$$

① Here again we choose the simplest possible element. If we had chosen $dA = dx\,dy$, we would have to integrate $y^2\,dx\,dy$ with respect to x first. This gives us $y^2 x\,dy$, which is the expression we chose at the outset.

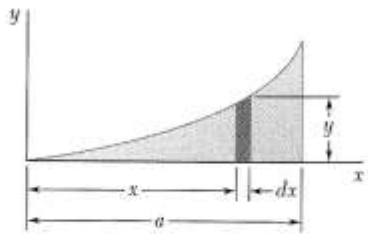
② Expressing x in terms of y should cause no difficulty if we observe the proportional relationship between the similar triangles.

Example 3:

(*a*) Determine the moment of inertia of the shaded area shown with respect to each of the coordinate axes. (Properties of this area were considered in Sample Prob. 5.4.) (*b*) Using the results of part *a*, determine the radius of gyration of the shaded area with respect to each of the coordinate axes.



Solution:



Referring to Sample Prob. 5.4, we obtain the following expressions for the equation of the curve and the total area:

$$y = \frac{b}{a^2}x^2 \qquad A = \tfrac{1}{3}ab$$

Moment of Inertia I_x. A vertical differential element of area is chosen to be dA. Since all portions of this element are *not* at the same distance from the x axis, we must treat the element as a thin rectangle. The moment of inertia of the element with respect to the x axis is then

$$dI_x = \tfrac{1}{3}y^3\,dx = \frac{1}{3}\left(\frac{b}{a^2}x^2\right)^3 dx = \frac{1}{3}\frac{b^3}{a^6}x^6\,dx$$

$$I_x = \int dI_x = \int_0^a \frac{1}{3}\frac{b^3}{a^6}x^6\,dx = \left[\frac{1}{3}\frac{b^3}{a^6}\frac{x^7}{7}\right]_0^a$$

$$I_x = \frac{ab^3}{21} \quad ◀$$

Moment of Inertia I_y. The same vertical differential element of area is used. Since all portions of the element are at the same distance from the y axis, we write

$$dI_y = x^2\,dA = x^2(y\,dx) = x^2\left(\frac{b}{a^2}x^2\right)dx = \frac{b}{a^2}x^4\,dx$$

$$I_y = \int dI_y = \int_0^a \frac{b}{a^2}x^4\,dx = \left[\frac{b}{a^2}\frac{x^5}{5}\right]_0^a$$

$$I_y = \frac{a^3b}{5} \quad ◀$$

Radii of Gyration k_x and k_y. We have, by definition,

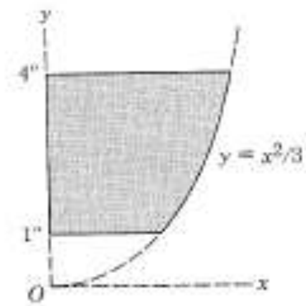
$$k_x^2 = \frac{I_x}{A} = \frac{ab^3/21}{ab/3} = \frac{b^2}{7} \qquad k_x = \sqrt{\tfrac{1}{7}}\,b \quad ◀$$

and

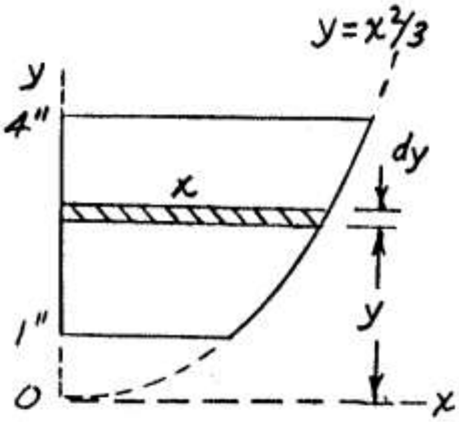
$$k_y^2 = \frac{I_y}{A} = \frac{a^3b/5}{ab/3} = \tfrac{3}{5}a^2 \qquad k_y = \sqrt{\tfrac{3}{5}}\,a \quad ◀$$

Example 4:

Calculate the moment of inertia of the shaded area about the y-axis.



Solution:



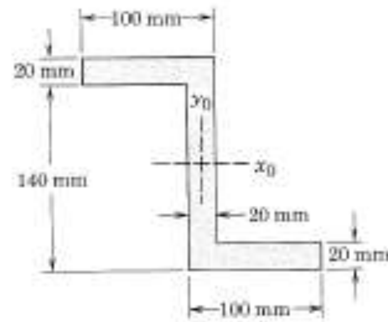
$$dI_y = \frac{1}{3}x^3\,dy = \frac{1}{3}(3y)^{3/2}\,dy = \sqrt{3}\,y^{3/2}\,dy$$

$$I_y = \sqrt{3}\int_1^4 y^{3/2}\,dy = \sqrt{3}\,\frac{2}{5}\left(4^{5/2} - 1^{5/2}\right) = \frac{2\sqrt{3}}{5}(32-1) = \underline{21.5 \text{ in.}^4}$$

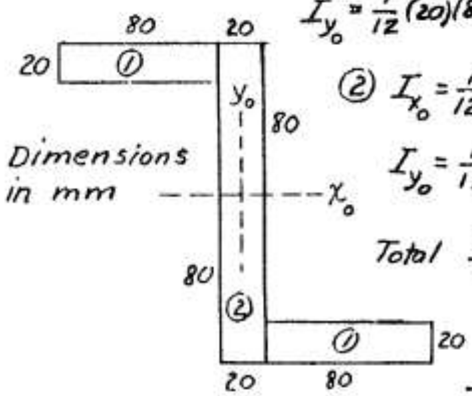
Composite bodies

Example 1:

Determine the moments of inertia of the Z-section about its centroidal x_0- and y_0-axes.



Solution:



$$① \; I_{x_0} = \frac{1}{12}(80)(20)^3 + (80)(20)(70)^2 = 7.89(10^6)\ mm^4$$

$$I_{y_0} = \frac{1}{12}(20)(80)^3 + (20)(80)(50)^2 = 4.85(10^6)\ mm^4$$

$$② \; I_{x_0} = \frac{1}{12}(20)(160)^3 = 6.83(10^6)\ mm^4$$

$$I_{y_0} = \frac{1}{12}(160)(20)^3 = 0.1067(10^6)\ mm^4$$

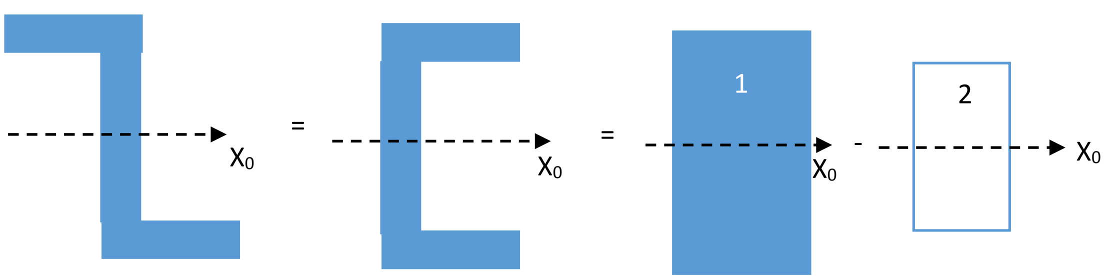
$$\text{Total } \bar{I}_x = [2(7.89) + 6.83](10^6)$$

$$= \underline{22.6(10^6)\ mm^4}$$

$$\bar{I}_y = [2(4.85) + 0.1067](10^6)$$

$$= \underline{9.81(10^6)\ mm^4}$$

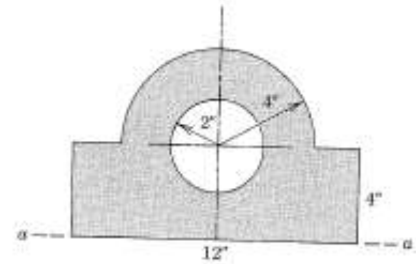
Alternatively for I_{x_0}:



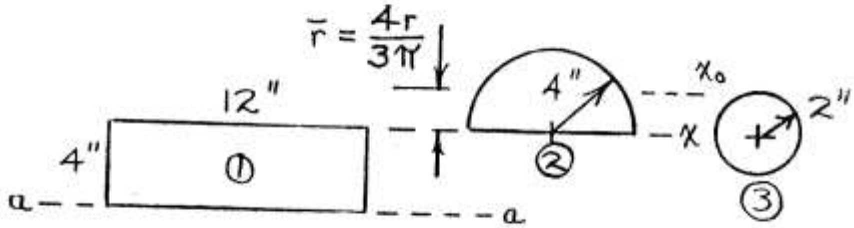
$$I_{x_0} = I_{1\,x_0} - I_{2\,x_0} = \frac{1}{12}\left[bh^3 - b'h'^3\right] = \frac{1}{12}[100 * 160^3\ mm^4 - 80 * 120^3\ mm^4] = 22{,}613{,}333\ mm^4$$

Example 2:

The cross section of a bearing block is shown in the figure by the shaded area. Calculate the moment of inertia of the section about its base *a-a*.



Solution:



Part 1 : $I_{a\text{-}a} = \frac{1}{3}(12)4^3 = 256 \text{ in.}^4$

Part 2 : $I_{a\text{-}a} = I_{x_o} + A\left(4 + \frac{4 \cdot 4}{3\pi}\right)^2$

where $I_{x_o} = I_x - A\bar{r}^2 = \frac{1}{8}\pi 4^4 - \pi \frac{4^2}{2}\left(\frac{4 \cdot 4}{3\pi}\right)^2$

$= 28.1 \text{ in.}^4$

So $I_{a\text{-}a} = 28.1 + \frac{\pi 4^2}{2}(32.5) = 844 \text{ in.}^4$

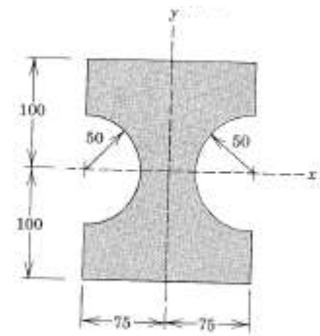
Part 3 : $I_{a\text{-}a} = I_x + A(4)^2 = \frac{1}{4}\pi 2^4 + \pi 2^2 (4)^2$

$= 214 \text{ in.}^4$

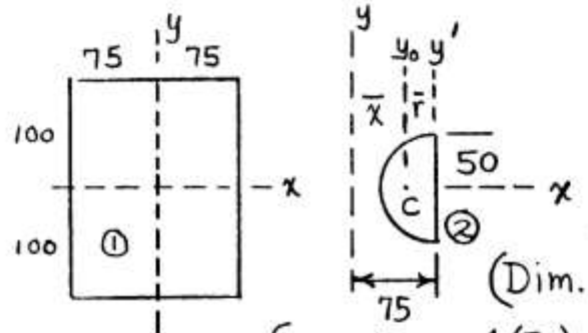
Combined : $I_{a\text{-}a} = 256 + 844 - 214 = \underline{886 \text{ in.}^4}$

Example 3:

Calculate the moments of inertia of the shaded area about the x- and y-axes.



Solution:



(Dim. in mm)

$$\begin{cases} \bar{r} = \dfrac{4r}{3\pi} = \dfrac{4(50)}{3\pi} = 21.2\ \text{mm} \\ \bar{x} = 75 - 21.2 = 53.8\ \text{mm} \end{cases}$$

Part I: $I_x = \frac{1}{12}(150)(200)^3 = 100(10^6)\ \text{mm}^4$

$I_y = \frac{1}{12}(200)(150)^3 = 56.2(10^6)\ \text{mm}^4$

Parts II: $I_x = \frac{1}{4}\pi(50)^4 = 4.91(10^6)\ \text{mm}^4$ (for both together)

$$I_y = I_{y_0} + A\bar{x}^2 = I_{y'} - A\bar{r}^2 + A\bar{x}^2 = I_{y'} + A(\bar{x}^2 - \bar{r}^2)$$

$$= \frac{1}{2}\left(\frac{1}{4}\pi\, 50^4\right) + \frac{\pi(50)^2}{2}(53.8^2 - 21.2^2)$$

$= 12.04(10^6)\ \text{mm}^4$ for each, $24.1(10^6)\ \text{mm}^4$ for both

Combined: $I_x = 100(10^6) - 4.91(10^6) = \underline{95.1(10^6)\ \text{mm}^4}$

$I_y = 56.2(10^6) - 24.1(10^6) = \underline{32.2(10^6)\ \text{mm}^4}$